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CURRENT ALGEBRAS AND CATEGORIFIED QUANTUM GROUPS

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Abstract. We identify the trace, or 0th Hochschild homology, of type ADE categorified quantum groups with the corresponding current algebra of the same type. To prove this, we show that 2-representations defined using categories of modules over cyclotomic (or deformed cyclotomic) quotients of KLR-algebras correspond to local (or global) Weyl modules. We also investigate the implications for centers of categories in 2-representations of categorified quantum groups.

Contents

1. Introduction

One very powerful idea in mathematics is categorification, and its necessary partner decategorification. Most work in recent years has understood decategorification to mean taking the Grothendieck group, but there are other ways of intepreting this idea. The one we will consider in this paper is the notion of **trace**.

The trace $\mathrm{Tr}(\mathcal{C})$ of a $\Bbbk$-linear category $\mathcal{C}$ is $\Bbbk$-vector space given by

$$\mathrm{Tr}(\mathcal{C}) = \left(\bigoplus_{x \in \mathrm{Ob}(\mathcal{C})} \mathrm{End}_{\mathcal{C}}(x) \right) / \, \mathrm{Span}_{\Bbbk}\{fg - gf\},$$

where f and g run through all pairs of morphisms $f\colon\ x \to y$, $g\colon\ y \to x$ with $x, y \in \mathrm{Ob}(\mathcal{C})$.

Trace (or 0th Hochschild homology) and Grothendieck group are closely related: there is a **Chern character map**

$$h_{\mathcal{C}} \colon K_0(\mathcal{C}) \to \mathrm{Tr}(\mathcal{C})$$

sending the class of an object to the image of its identity morphism in the trace. The interplay of these two kinds of decategorification has shown up in many contexts, most classically in the various

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generalizations of the Riemann-Roch theorem. See [9] for a more categorical perspective and [3] for more general discussion by the first three authors and Guliyev.

In a certain sense, trace decategorification is richer than Grothendieck decategorification. The Chern character map is usually injective, but often fails to be surjective, so there are many classes in the trace which do not correspond to any object. To use an extremely loose analogy, the Grothendieck group is something like H_0 of a space, and the trace or Hochschild homology like its homology. In fact, the Borel-Moore homology of a space X can be interpreted as the Hochschild homology of the category of constructible sheaves on X.

We'll concentrate on only one small aspect of this large topic. For any 2-category $\mathbf{C}$, we can consider its trace, as defined in Section 2.6. This trace is naturally a category. A 2-representation of $\mathbf{C}$ is a 2-functor to Cat[1], the 2-category of categories, functors and natural transformations. As explained in Section 2.7 the 2-representation R induces a representation of the 1-category $\mathrm{Tr}(\mathbf{C})$ sending each object c to the trace of the category $R(c)$.

Actually, let us specialize this yet further: the 2-category $\mathbf{C}$ that we'll consider is the categorified quantum group $\mathcal{U}^* = \mathcal{U}^*_Q(\mathfrak{g})$ attached to a symmetric Kac-Moody Lie algebra $\mathfrak{g}$ [25]. This is a 2-category with many interesting aspects; for us the most important is its "representation theory." For each highest weight λ, there are two representation categories $\mathcal{U}^\lambda$, and $\check{\mathcal{U}}^\lambda$, with Grothendieck groups that agree with the simple highest weight representation $V(\lambda)$ of $\mathfrak{g}$. The category $\mathcal{U}^\lambda$ is the category of projective modules over the cyclotomic quotient as introduced in [24, §3.4] and $\check{\mathcal{U}}^\lambda$ the category of projective modules over the deformed cyclotomic quotient defined in [47, 38].

When we take the Grothendieck group of the Karoubi completion of $\mathcal{U}^*$, we obtain the category $\dot{\mathbf{U}}(\mathfrak{g})$, which is the idempotented version of the universal enveloping algebra $\mathbf{U}(\mathfrak{g})$; the trace will prove to be quite a bit larger. Our study of the trace was motivated by geometric considerations. For each highest weight λ, there is collection of quiver varieties, constructed by Nakajima [36]. The quiver varieties and 2-category $\mathcal{U}^*$ are closely related; $\mathcal{U}^*$ acts in a natural way on the (quantum) coherent sheaves on these varieties [11, 43], and many constructions which first appeared in one context have analogs in the other (for example, Lusztig's canonical basis appears naturally in both).

This philosophy suggests that the trace of the category $\mathcal{U}^*$ should be connected to the homology of quiver varieties[2]. Work of Varagnolo shows that there is an action of the current algebra $\mathbf{U}(\mathfrak{g}[\mathbf{t}])$ on the whole Borel-Moore homology (or cohomology) of the quiver varieties for λ, identifying their sum with the Weyl module of highest weight λ over the current algebra (or dual Weyl module). In this paper we'll discuss the analogue of Varagnolo's construction in our algebraic context, which is given by the trace decategorification we've discussed.

Theorem A. *Assume $\mathfrak{g}$ is type ADE. Then $\mathrm{Tr}(\mathcal{U}^\lambda)$ is isomorphic to the local Weyl module of highest weight λ, and $\mathrm{Tr}(\check{\mathcal{U}}^\lambda)$ is isomorphic to the global Weyl module. Dually, the center of $\mathcal{U}^\lambda$ is isomorphic to the dual local Weyl module.*

In fact, this result has recently been shown independently by Shan, Varagnolo, and Vasserot [41]; their techniques are quite similar to ours, having been inspired by the same geometric considerations.

Our motivation in studying traces was to identify the trace of the 2-category $\mathcal{U}^*$ itself. Here we prove the following theorem:

Theorem B. *Assume $\mathfrak{g}$ is type ADE. The trace category $\mathrm{Tr}(\mathcal{U}^*)$ is canonically isomorphic to the category $\dot{\mathbf{U}}(\mathfrak{g}[t])$. The isomorphisms of Theorem A are induced by this isomorphism.*

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[1] More generally, we can speak of a 2-representation of $\mathbf{C}$ in an arbitrary bicategory $\mathcal{D}$, such as the bicategory of rings, bimodules, and bimodule homomorphisms. This is a 2-functor $\mathbf{C} \to \mathcal{D}$.

[2] The reader who is paying attention here might rightly complain "Shouldn't it be connected to the Hochschild homology of $\mathcal{U}^*$?" Actually, in the geometric context, the grading on $\mathcal{U}^*$ becomes homological in nature, and what we think of as the trace is really part of the Hochschild homology.

This result intimately links the study of 2-representations of $\mathcal{U}^*(\mathfrak{g})$ with the representation theory of the current algebra $\mathbf{U}(\mathfrak{g}[\mathbf{t}])$. As explained above, any 2-representation of $\mathcal{U}^*$ gives rise to a representation of $\mathrm{Tr}(\mathcal{U}^*)$ and hence the current algebra.

The 2-category $\mathcal{U}^*(\mathfrak{g})$ is known to act on numerous categories of interest including:

- categories of modules over the symmetric group [16, 7],
- parabolic category $\mathcal{O}$ for $\mathfrak{gl}_N$ [8, 47, 21],
- derived categories of coherent sheaves on Nakajima quiver varieties [11, 39],
- coherent sheaves on certain convolution varieties obtained from the affine Grassmannian [10],
- categorified Fock space representations of certain Heisenberg algebras [13, 10],
- category $\mathcal{O}$ for a rational Cherednik algebra of $G(n, 1, r)$ [40],
- categories of $\mathfrak{sl}_n$-foams used in link homology theory [33, 29, 37], and
- categories of $\mathfrak{sl}_n$-matrix factorizations [34].

Theorem B indicates that all of these 2-representations give rise to current algebra representations. As discussed earlier, there is Chern map relating the Grothendieck and trace decategorifications, and the induced map commutes with the $\mathfrak{g}$-action; surprisingly, Theorem A shows that that there are 2-representations with the same Grothendieck decategorification can have different trace decategorifications.

Finally, another natural construction on categories is the notion of the center of an additive category $Z(\mathcal{C})$. This is defined as the commutative monoid of endo-natural transformations of the identity functor $\mathrm{End}(\mathrm{Id}_{\mathcal{C}})$. The center and trace of a category are closely related. Here we also prove the following:

Theorem C. *In general simply-laced type, any 2-representation of $\mathcal{U}^*$ into the 2-category of additive $\Bbbk$-linear categories gives rise to an action of the current algebra $\mathrm{Tr}(\mathcal{U}^*)$ on the centers of the categories defining the 2-representation.*

In particular, the current algebra acts on the centers of all of the categories listed above. A special case of this fact was already observed by Brundan. He made the surprising discovery that one could define an action of the Lie algebra $\widehat{\mathfrak{g}} := \mathfrak{gl}_\infty(\mathbb{C})$ on the center $Z(\mathcal{O}) = \bigoplus_\nu Z(\mathcal{O}_\nu)$ of all integral blocks $\mathcal{O}_\nu$ of category $\mathcal{O}$ for $\mathfrak{gl}_n$ [6]. In this action, the Chevalley generators of $\widehat{\mathfrak{g}}$ act as certain trace maps associated to canonical adjunction maps between special translation functors that arise from tensoring with a $\mathfrak{g}$-module and its dual. Theorem C gives a new construction of Brundan's action as well as extends it to an action of the current algebra associated to $\widehat{\mathfrak{g}}$.

The paper is organized as follows: In the Sections 2–4, we present some general facts about the trace and define different versions of the categorified quantum groups and current algebra. In Section 5, we define the map of the current algebra to the trace. Finally, we prove Theorem A, using results from Theorems 7.3 and 7.4 and Corollary 9.4. Theorem B is equivalent to Theorem 8.3. Theorem C is proven in the last section where rescaling 2-functors needed to make $\mathcal{U}^*(\mathfrak{g})$ cyclic are also studied.

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2. The trace decategorification map

2.1. **Traces of linear categories.** In what follows, we recall the notion of the trace of linear categories and generalizations. For more details, see e.g. [4, 3]. We focus on linear categories over a fixed field $\Bbbk$.

Let $\mathcal{C}$ be a small $\Bbbk$-linear category. Thus, the hom spaces $\mathcal{C}(x, y)$ are $\Bbbk$-vector spaces, and the composition is bilinear over $\Bbbk$.

Define the *trace* $\mathrm{Tr}(\mathcal{C})$ of $\mathcal{C}$, also known as the zeroth Hochschild-Mitchell homology $\mathrm{HH}_0(\mathcal{C})$, by

$$\mathrm{Tr}(\mathcal{C}) = \left(\bigoplus_{x \in \mathrm{Ob}(\mathcal{C})} \mathrm{End}_{\mathcal{C}}(x) \right) / \operatorname{Span}_{\Bbbk}\{fg - gf\},$$

where f and g runs through all pairs of morphisms $f\colon x \to y$, $g\colon y \to x$ with $x, y \in \mathrm{Ob}(\mathcal{C})$. (Note that $\mathrm{Tr}(\mathcal{C})$ depends on the base field $\Bbbk$, but we usually omit it in the notation.) For $f\colon x \to x$, let $[f] \in \mathrm{Tr}(\mathcal{C})$ denote the corresponding equivalence class.

Recall that a $\Bbbk$-linear category with one object is identified with a $\Bbbk$-algebra. For a $\Bbbk$-algebra A, we set

$$\mathrm{Tr}(A) = \mathrm{HH}_0(A) = A/[A, A] = A/\operatorname{Span}_{\Bbbk}\{ab - ba \mid a, b \in A\}.$$

The trace Tr gives a functor from the small $\Bbbk$-linear categories to the $\Bbbk$-vector spaces. If $F\colon \mathcal{C} \to \mathcal{D}$ is a $\Bbbk$-linear functor, then F induces a linear map on traces

$$\mathrm{Tr}(F)\colon\ \mathrm{Tr}(\mathcal{C}) \to \mathrm{Tr}(\mathcal{D})$$

given by $\mathrm{Tr}(F)([f]) = [F(f)]$ for endomorphisms $f\colon x \to x$ in $\mathcal{C}$. Furthermore, if $\alpha\colon F \Rightarrow F\colon \mathcal{C} \to \mathcal{D}$ is a natural transformation of $\Bbbk$-linear functors, then α gives rise to a linear map

$$\begin{aligned} \mathrm{Tr}(\alpha)\colon\ & \mathrm{Tr}(\mathcal{C}) \to \mathrm{Tr}(\mathcal{D}) \\ & [f\colon x \to x] \mapsto [\alpha_x \circ F(f)]. \end{aligned} \tag{2.1}$$

Lemma 2.1. Let $\mathcal{C}$ be a $\Bbbk$-linear additive category. Let $S \subset \mathrm{Ob}(\mathcal{C})$ be a subset such that every object in $\mathcal{C}$ is isomorphic to the direct sum of finitely many copies of objects in S. Let $\mathcal{C}|_S$ denote the full subcategory of $\mathcal{C}$ with $\mathrm{Ob}(\mathcal{C}|_S) = S$. Then, the inclusion functor $\mathcal{C}|_S \to \mathcal{C}$ induces an isomorphism

$$\mathrm{Tr}(\mathcal{C}|_S) \cong \mathrm{Tr}(\mathcal{C}) \tag{2.2}$$

Proof. The inclusion functor $\mathcal{C}|_S \to \mathcal{C}$ factors, uniquely up to natural isomorphisms, as

$$\mathcal{C}|_S \to (\mathcal{C}|_S)^{\oplus} \simeq \mathcal{C},$$

where $(\mathcal{C}|_S)^{\oplus}$ is the additive closure of $\mathcal{C}|_S$. Here $\mathcal{C}|_S \to (\mathcal{C}|_S)^{\oplus}$ is the canonical functor, and $(\mathcal{C}|_S)^{\oplus} \simeq \mathcal{C}$ is the canonical equivalence. These functors induce isomorphisms on Tr

$$\mathrm{Tr}(\mathcal{C}|_S) \cong \mathrm{Tr}((\mathcal{C}|_S)^{\oplus}) \cong \mathrm{Tr}(\mathcal{C}).$$

□

2.2. **Split Grothendieck groups and Chern character.** For a $\Bbbk$-linear additive category $\mathcal{C}$, the *split Grothendieck group* $K_0(\mathcal{C})$ of $\mathcal{C}$ is the abelian group generated by the isomorphism classes of objects of $\mathcal{C}$ with relations $[x \oplus y]_{\cong} = [x]_{\cong} + [y]_{\cong}$ for $x, y \in \mathrm{Ob}(\mathcal{C})$. Here $[x]_{\cong}$ denotes the isomorphism class of x.

For a $\Bbbk$-linear additive category $\mathcal{C}$, the *Chern character* for $\mathcal{C}$ is the $\Bbbk$-linear map

$$h_{\mathcal{C}}\colon\ K_0^{\Bbbk}(\mathcal{C}) := K_0(\mathcal{C}) \otimes \Bbbk \longrightarrow \mathrm{Tr}(\mathcal{C})$$

defined by $h_{\mathcal{C}}([x]_{\cong}) = [1_x]$ for $x \in \mathrm{Ob}(\mathcal{C})$. (Although $h_{\mathcal{C}}$ can be defined on $K_0(\mathcal{C})$, we consider only the above $\Bbbk$-linear version for simplicity.)

2.3. **Chern character for Krull-Schmidt categories.** A $\Bbbk$-linear additive category $\mathcal{C}$ is said to be *Krull-Schmidt* if every object in $\mathcal{C}$ decomposes in a unique way as the direct sum of finitely many indecomposable objects with local endomorphism rings. In a Krull-Schmidt category,

- an object is indecomposable if and only if its endomorphism ring is local,
- idempotents split (see Subsection 2.5 below).

Let $\mathcal{C}$ be a $\Bbbk$-linear Krull-Schmidt category. Fix a subset $I \subset \mathrm{Ob}(\mathcal{C})$ consisting of exactly one from each isomorphism class of indecomposable objects in $\mathcal{C}$. Then the split Grothendieck group $K_0(\mathcal{C})$ is a free abelian group with basis given by the isomorphism classes of indecomposable objects in $\mathcal{C}$. Hence we have

$$K_0^{\Bbbk}(\mathcal{C}) \cong \Bbbk \cdot I = \bigoplus_{x \in I} \Bbbk. \tag{2.3}$$

Let $\mathcal{J}$ be the two-sided ideal in $\mathcal{C}|_I$ generated by

- $J(\mathcal{C}(x,x))$ for $x \in I$, and
- $\mathcal{C}(x,y)$ for $x, y \in I$, $x \neq y$.

where $J(\mathcal{C}(x,x))$ is the Jacobson radical of the endomorphism ring $\mathcal{C}(x,x)$. (I.e., $\mathcal{J}$ is the smallest family of $\Bbbk$-subspaces $\mathcal{J}(x,y)$ for $x, y \in I$ which contains the subspaces given above and is closed under left and right composition with morphisms in $\mathcal{C}|_I$.)

In fact, $\mathcal{J}$ coincides the Jacobson radical of the linear category $\mathcal{C}|_I$, defined in [23, 32].

Lemma 2.2. For $x \in I$, we have

$$\mathcal{J}(x,x) = J(\mathcal{C}(x,x)), \tag{2.4}$$

Proof. We have

$$\mathcal{J}(x,x) = J(\mathcal{C}(x,x)) + \sum_{y \in I, y \neq x} \mathcal{C}(y,x) \circ \mathcal{C}(x,y).$$

Therefore, it suffices to show that, for $x, y \in I$ with $x \neq y$, we have $\mathcal{C}(y,x) \circ \mathcal{C}(x,y) \subset J(\mathcal{C}(x,x))$.

We will show that if $f\colon x \to y$, $g\colon y \to x$, then $gf \in J(\mathcal{C}(x,x))$. Suppose $gf \notin J(\mathcal{C}(x,x))$ for contradiction. Since $\mathcal{C}(x,x)$ is local, it follows that gf is an isomorphism. Hence $f(gf)^{-1}g \in \mathcal{C}(y,y)$ is an idempotent. Since $\mathcal{C}$ is Krull-Schmidt, it follows that y has a direct summand isomorphic to x. Since y is indecomposable, it follows that $y \cong x$. This is a contradiction. $\square$

By Lemma 2.2, the quotient category $(\mathcal{C}|_I)/\mathcal{J}$ has the following hom spaces.

$$((\mathcal{C}|_I)/\mathcal{J})(x,y) = \begin{cases} D_x := \mathcal{C}(x,x)/J(\mathcal{C}(x,x)), & x = y, \\ 0, & x \neq y. \end{cases} \tag{2.5}$$

Note that D_x is a division algebra over $\Bbbk$. By (2.5), we have

$$\mathrm{Tr}((\mathcal{C}|_I)/\mathcal{J}) \cong \bigoplus_{x \in I} D_x/[D_x, D_x].$$

Let $\eta\colon \bigoplus_{x\in I} \Bbbk \to \bigoplus_{x \in I} D_x$ be the composite

$$\bigoplus_{x \in I} \Bbbk \underset{(2.3)}{\cong} K_0^{\Bbbk}(\mathcal{C}) \xrightarrow{h_{\mathcal{C}}} \mathrm{Tr}(\mathcal{C}) \underset{(2.2)}{\cong} \mathrm{Tr}(\mathcal{C}|_I) \twoheadrightarrow \mathrm{Tr}((\mathcal{C}|_I)/\mathcal{J}) \cong \bigoplus_{x \in I} D_x/[D_x, D_x], \tag{2.6}$$

where $\twoheadrightarrow$ is induced by the projection $\mathcal{C}|_I \to (\mathcal{C}|_I)/\mathcal{J}$. It is easy to check that $\eta = \bigoplus_{x \in I} \eta_x$, where $\eta_x\colon \Bbbk \to D_x/[D_x, D_x]$ is defined by $\eta_x(1_{\Bbbk}) = [1_{D_x}]$. By (2.6), we have the following.

Lemma 2.3. If η_x is injective (i.e. $1 \notin [D_x, D_x]$) for all $x \in I$, then $h_{\mathcal{C}}$ is injective.

Using Lemma 2.3, we prove the following.

Proposition 2.4. Let $\Bbbk$ be a perfect field. Let $\mathcal{C}$ be a $\Bbbk$-linear Krull-Schmidt category with finite dimensional endomorphism algebra for each indecomposable object. Then, the Chern character map $h_{\mathcal{C}}\colon K_0(\mathcal{C})\otimes_{\mathbb{Z}}\Bbbk \to \mathrm{Tr}(\mathcal{C})$ is injective.

Proof. By Lemma 2.3, it suffices to prove that for each $x\in I$, $1_{D_x}\notin [D_x,D_x]$.

Note that D_x is a finite dimensional division algebra over $\Bbbk$. Let K be the center of D_x, which is a finite extension field of $\Bbbk$.

For $u\in D_x$, let $L_u\colon D_x\to D_x$ be left multiplication by u, which is a K-linear map. Define a K-linear map $\tau_x\colon D_x\to K$ by

$$\tau_x(u)=\mathrm{tr}(L_u).$$

We have $\tau_x([D_x,D_x])=0$, since

$$\mathrm{tr}(L_{[u,v]})=\mathrm{tr}([L_u,L_v])=0.$$

We have

$$\tau_x(1_{D_x})=\mathrm{tr}(L_{1_{D_x}})=\dim_K D_x.$$

If $\Bbbk$ is of characteristic $p>0$, then since $\Bbbk$ is perfect, it follows from [1, Theorem 13] that $\dim_K D_x$ is not divisible by p. Hence it follows that $1_{D_x}\notin [D_x,D_x]$, regardless of the characteristic of $\Bbbk$. $\square$

Proposition 2.5. Let $\Bbbk$ be a field, and let $\mathcal{C}$ be a $\Bbbk$-linear Krull-Schmidt category such that for each indecomposable object x we have $\mathcal{C}(x,x)/J(\mathcal{C}(x,x))\cong\Bbbk$. (This condition holds when $\Bbbk$ is algebraically closed and each $\mathcal{C}(x,x)$ is finite dimensional.) Then the Chern character $h_{\mathcal{C}}$ is split injective with a unique splitting

$$p_{\mathcal{C}}\colon\ \mathrm{Tr}(\mathcal{C})\to K_0^{\Bbbk}(\mathcal{C})$$

such that, for $x\in I$, $p_{\mathcal{C}}([1_x])=[x]_{\cong}$ and $p_{\mathcal{C}}([f])=0$ for $f\in J(\mathcal{C}(x,x))$.

Proof. We have $D_x=D_x/[D_x,D_x]\cong\Bbbk$ for each $x\in I$. Using (2.6) and (2.3), we obtain the result. $\square$

2.4. **Graded categories.** A *graded* $\Bbbk$-linear category is a $\Bbbk$-linear category $\mathcal{C}$ equipped with an auto-equivalence $\langle 1\rangle\colon\mathcal{C}\xrightarrow{\simeq}\mathcal{C}$. For $t\geq 0$, set $\langle t\rangle=\langle 1\rangle^t$, and, for $t<0$, $\langle t\rangle=\langle -1\rangle^{-t}$, where $\langle -1\rangle\colon\mathcal{C}\xrightarrow{\simeq}\mathcal{C}$ is an inverse (unique up to natural isomorphism) of $\langle 1\rangle$.

The auto-equivalence $\langle 1\rangle$ induces $\Bbbk$-linear automorphisms

$$q=K_0^{\Bbbk}(\langle 1\rangle)\colon\ K_0^{\Bbbk}(\mathcal{C})\xrightarrow{\cong}K_0^{\Bbbk}(\mathcal{C}),$$
$$q=\mathrm{Tr}(\langle 1\rangle)\colon\ \mathrm{Tr}(\mathcal{C})\xrightarrow{\cong}\mathrm{Tr}(\mathcal{C}),$$

which give $K_0^{\Bbbk}(\mathcal{C})$ and $\mathrm{Tr}(\mathcal{C})$, respectively, $\Bbbk[q,q^{-1}]$-module structures. The Chern character map

$$h_{\mathcal{C}}\colon\ K_0^{\Bbbk}(\mathcal{C})\to\mathrm{Tr}(\mathcal{C})$$

is a $\Bbbk[q,q^{-1}]$-module homomorphism, since $h_{\mathcal{C}}$ is a natural transformation.

A *translation* in a graded $\Bbbk$-linear category $\mathcal{C}$ is a family of natural isomorphisms

$$x\xrightarrow{\cong}x\langle 1\rangle.$$

If a graded $\Bbbk$-linear category $\mathcal{C}$ admits a translation, then it makes the action q on $K_0^{\Bbbk}(\mathcal{C})$ and $\mathrm{Tr}(\mathcal{C})$ trivial. Thus, in this case, $K_0^{\Bbbk}(\mathcal{C})$ and $\mathrm{Tr}(\mathcal{C})$ are $\Bbbk$-vector spaces rather than $\Bbbk[q,q^{-1}]$-modules.

Given any graded $\Bbbk$-linear category $\mathcal{C}$, we can form a graded $\Bbbk$-linear category $\mathcal{C}^*$ with translation, such that $\mathrm{Ob}(\mathcal{C})=\mathrm{Ob}(\mathcal{C}^*)$ and

$$\mathcal{C}^*(x,y):=\bigoplus_{t\in\mathbb{Z}}\mathcal{C}(x,y\langle t\rangle),$$

for all $x, y \in \mathrm{Ob}(\mathcal{C})$. Note that $\mathcal{C}^*$ is equipped with a $\mathbb{Z}$-grading with the degree t hom space given by

$$\mathcal{C}^*_t(x, y) := \mathcal{C}(x, y\langle t\rangle), \quad t \in \mathbb{Z}.$$

Thus $\mathcal{C}^*$ is enriched over $\mathbb{Z}$-graded vector spaces. Hence, the trace $\mathrm{Tr}(\mathcal{C}^*)$ has a $\mathbb{Z}$-graded $\Bbbk$-vector space structure

$$\mathrm{Tr}(\mathcal{C}^*) = \bigoplus_{t\in\mathbb{Z}} \mathrm{Tr}_t(\mathcal{C}^*),$$

where $\mathrm{Tr}_t(\mathcal{C}^*)$ is spanned by $[f]$ for endomorphisms in $\mathcal{C}^*$ of degree t.

2.5. **Traces and the Karoubi envelope.** An idempotent $e\colon x \to x$ in $\mathcal{C}$ in a category $\mathcal{C}$ is said to *split* if there is an object y and morphisms $g\colon x \to y$, $h\colon y \to x$ such that $hg = e$ and $gh = 1_y$.

The *Karoubi envelope* $\mathrm{Kar}(\mathcal{C})$ (also called *idempotent completion*) of $\mathcal{C}$ is the category whose objects are pairs (x, e) of objects $x \in \mathrm{Ob}(\mathcal{C})$ and an idempotent endomorphism $e\colon x \to x$, $e^2 = e$, in $\mathcal{C}$ and whose morphisms

$$f\colon (x, e) \to (y, e')$$

are morphisms $f\colon x \to y$ in $\mathcal{C}$ such that $f = e'fe$. Composition is induced by the composition in $\mathcal{C}$ and the identity morphism is $e\colon (x, e) \to (x, e)$. $\mathrm{Kar}(\mathcal{C})$ is equipped with a linear category structure. It is well known that the Karoubi envelope of an additive category is additive.

There is a natural fully faithful linear functor

$$\iota\colon \mathcal{C} \to \mathrm{Kar}(\mathcal{C})$$

such that $\iota(x) = (x, 1_x)$ for $x \in \mathrm{Ob}(\mathcal{C})$ and $\iota(f\colon x \to y) = f$. The Karoubi envelope $\mathrm{Kar}(\mathcal{C})$ is universal in the sense that if $F\colon \mathcal{C} \to \mathcal{D}$ is a linear functor from $\mathcal{C}$ to a linear category $\mathcal{D}$ with split idempotents, then F extends to a functor from $\mathrm{Kar}(\mathcal{C})$ to $\mathcal{D}$ uniquely up to natural isomorphism [5, Proposition 6.5.9].

The functor $\iota\colon \mathcal{C} \longrightarrow \mathrm{Kar}(\mathcal{C})$ induces an isomorphism

$$\mathrm{Tr}(\iota)\colon \ \mathrm{Tr}(\mathcal{C}) \xrightarrow{\cong} \mathrm{Tr}(\mathrm{Kar}(\mathcal{C})), \tag{2.7}$$

so that the trace decategorification map is invariant under the passage to the Karoubi envelope.

2.6. **K_0, Tr and Kar for 2-categories.** We can extend many of the constructions defined above for (additive) $\Bbbk$-linear categories to the 2-categorical setting. A 2-category $\mathbf{C}$ is linear if the categories $\mathbf{C}(x, y)$ are linear for all $x, y \in \mathrm{Ob}(\mathbf{C})$ and the composition functor preserves the linear structure (see [5] for more details). Similarly, an additive linear 2-category is a linear 2-category in which the categories $\mathbf{C}(x, y)$ are also additive and composition is given by an additive functor.

The following definitions extend the Karoubi envelope, split Grothendieck group, and trace to the 2-categorical setting.

- Given an additive linear 2-category $\mathbf{C}$, define the split Grothendieck group $K_0(\mathbf{C})$ of $\mathbf{C}$ to be the linear category with $\mathrm{Ob}(K_0(\mathbf{C})) = \mathrm{Ob}(\mathbf{C})$ and with $K_0(\mathbf{C})(x, y) := K_0(\mathbf{C}(x, y))$ for any two objects $x, y \in \mathrm{Ob}(\mathbf{C})$. For $[f]_\cong \in \mathrm{Ob}(K_0(\mathbf{C})(x, y))$ and $[g]_\cong \in \mathrm{Ob}(K_0(\mathbf{C})(y, z))$ the composition in $K_0(\mathbf{C})$ is defined by $[g]_\cong \circ [f]_\cong := [g \circ f]_\cong$.
- For a 2-category $\mathbf{C}$, we define a category $\mathrm{Tr}(\mathbf{C})$ with $\mathrm{Ob}(\mathrm{Tr}(\mathbf{C})) = \mathrm{Ob}(\mathbf{C})$ as follows. For $x, y \in \mathrm{Ob}(\mathbf{C})$, set $\mathrm{Tr}(\mathbf{C})(x, y) = \mathrm{Tr}(\mathbf{C}(x, y))$. For $x, y, z \in \mathrm{Ob}(\mathbf{C})$, define composition for $\sigma \in \mathrm{End}_{\mathbf{C}(x,y)}$ so that $\tau \in \mathrm{End}_{\mathbf{C}(y,z)}$, we have $[\tau] \circ [\sigma] = [\tau \circ \sigma]$. The identity morphism for $x \in \mathrm{Ob}(\mathrm{Tr}(\mathbf{C})) = \mathrm{Ob}(\mathbf{C})$ is given by $[1_{I_x}]$.
- The Karoubi envelope $\mathrm{Kar}(\mathbf{C})$ of an additive $\Bbbk$-linear 2-category $\mathbf{C}$ is the linear 2-category with $\mathrm{Ob}(\mathrm{Kar}(\mathbf{C})) = \mathrm{Ob}(\mathbf{C})$ and with hom categories $\mathrm{Kar}(\mathbf{C})(x, y) := \mathrm{Kar}(\mathbf{C}(x, y))$. The composition functor $\mathrm{Kar}(\mathbf{C})(y, z) \times \mathrm{Kar}(\mathbf{C})(x, y) \to \mathrm{Kar}(\mathbf{C})(x, z)$ is induced by the universal property of the Karoubi envelope from the composition functor in $\mathbf{C}$. The fully-faithful additive

functors $\mathbf{C}(x,y) \to \mathrm{Kar}(\mathbf{C}(x,y))$ combine to form an additive 2-functor $\mathbf{C} \to \mathrm{Kar}(\mathbf{C})$ that is universal with respect to splitting idempotents in the Hom-categories $\mathbf{C}(x,y)$.

A $\Bbbk$-*linear 2-category* is a 2-category $\mathbf{C}$ such that

(1) for $x, y \in \mathrm{Ob}(\mathbf{C})$, the category $\mathbf{C}(x,y)$ is equipped with a structure of a $\Bbbk$-linear category,
(2) for $x, y, z \in \mathrm{Ob}(\mathbf{C})$, the functor $\circ\colon \mathbf{C}(y,z) \times \mathbf{C}(x,y) \to \mathbf{C}(x,z)$ is "bilinear" in the sense that the functors $- \circ f\colon \mathbf{C}(y,z) \to \mathbf{C}(x,z)$ for $f \in \mathrm{Ob}(\mathbf{C}(x,y))$ and $g \circ -\colon \mathbf{C}(x,y) \to \mathbf{C}(x,z)$ for $g \in \mathrm{Ob}(\mathbf{C}(y,z))$ are $\Bbbk$-linear functors.

The trace $\mathrm{Tr}(\mathbf{C})$ of a linear 2-category $\mathbf{C}$ is defined similarly to the trace of 2-category, and is equipped with a linear category structure.

The homomorphisms $h_{\mathbf{C}(x,y)}$ taken over all objects $x, y \in \mathrm{Ob}(\mathbf{C})$ give rise to a $\Bbbk$-linear functor

$$h_{\mathbf{C}}\colon \mathcal{K}_0(\mathbf{C}) \to \mathrm{Tr}(\mathbf{C}) \tag{2.8}$$

which is the identity map on objects and sends $K_0(\mathbf{C})(x,y) \to \mathrm{Tr}(\mathbf{C})(x,y)$ via the homomorphism $h_{\mathbf{C}(x,y)}$. It is easy to see that this assignment preserves composition since

$$h_{\mathbf{C}}([g]_{\cong} \circ [f]_{\cong}) = h_{\mathbf{C}}([g \circ f]_{\cong}) = [1_{g\circ f}] = [1_g \circ 1_f] = [1_g] \circ [1_f] = h_{\mathbf{C}}([g]_{\cong}) \circ h_{\mathbf{C}}([f]_{\cong}). \tag{2.9}$$

2.7. **2-functoriality of** Tr **on linear 2-categories.** A (strict) 2-functor $F\colon \mathbf{C} \to \mathbf{D}$ between linear 2-categories $\mathbf{C}$ and $\mathbf{D}$ is a *linear 2-functor* if for $x, y \in \mathrm{Ob}(\mathbf{C})$ the functor $F\colon \mathbf{C}(x,y) \to \mathbf{D}(x,y)$ is linear. Then F induces a linear functor

$$\mathrm{Tr}(F)\colon\ \mathrm{Tr}(\mathbf{C}) \to \mathrm{Tr}(\mathbf{D})$$

such that the map $F\colon\ \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$ on objects gives the map

$$\mathrm{Tr}(F) = F\colon\ \mathrm{Ob}(\mathrm{Tr}(\mathbf{C})) \to \mathrm{Ob}(\mathrm{Tr}(\mathbf{D})),$$

and, for $x, y \in \mathrm{Ob}(\mathbf{C})$, the linear functor $F_{x,y}\colon\ \mathbf{C}(x,y) \to \mathbf{D}(F(x),F(y))$ induces the linear map

$$\mathrm{Tr}(F)_{x,y} = \mathrm{Tr}(F_{x,y})\colon\ \mathrm{Tr}(\mathbf{C})(x,y) \to \mathrm{Tr}(\mathbf{D})(F(x),F(y)).$$

It is possible to work more generally in the context of linear bicategories and non-strict 2-functors, however this generality is not needed here.

In the case when $\mathbf{D} = \mathbf{LinCat}$, the 2-category of $\Bbbk$-linear categories, $\Bbbk$-linear functors, and $\Bbbk$-linear natural transformations, a 2-functor $F\colon \mathbf{C} \to \mathbf{LinCat}$ can be used to define a representation

$$\rho_F\colon\ \mathrm{Tr}(\mathbf{C}) \to \mathbf{Vect}_{\Bbbk} \tag{2.10}$$

sending each object x of $\mathrm{Tr}(\mathbf{C})$ to the $\Bbbk$-vector space $\mathrm{Tr}(F(x))$, and each morphism $[\sigma]\colon x \to y$ in $\mathrm{Tr}(\mathbf{C})$, with $\sigma\colon f \Rightarrow f\colon x \to y$ in $\mathbf{C}$, to the linear map

$$\rho_F([\sigma])\colon\ \mathrm{Tr}(F(x)) \longrightarrow \mathrm{Tr}(F(y)), \tag{2.11}$$

such that for $[g\colon u \to u] \in \mathrm{Tr}(F(x))$ we have

$$\rho_F([\sigma])([g]) = [F(f)(g) \circ F(\sigma)_u] = [F(\sigma)_u \circ F(f)(g)].$$

Here, note that $F(\sigma)\colon F(f) \Rightarrow F(f)\colon F(x) \to F(y)$ is a natural transformation. Hence, using equation (2.1) we see that (2.11) is well-defined.

3. The current algebra $\mathbf{U}_q(\mathfrak{g}[t])$

3.1. The quantum group $\mathbf{U}_q(\mathfrak{g})$.

3.1.1. *Cartan data.* For this article we restrict our attention to simply-laced Kac-Moody algebras. These algebras are associated to a symmetric Cartan data consisting of

- a free $\mathbb{Z}$-module X (the weight lattice),
- for $i \in I$ (I is an indexing set) there are elements $\alpha_i \in X$ (simple roots) and $\Lambda_i \in X$ (fundamental weights),
- for $i \in I$ an element $h_i \in X^\vee = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ (simple coroots),
- a bilinear form $(\cdot,\cdot)$ on X.

Write $\langle \cdot,\cdot\rangle \colon X^\vee \times X \to \mathbb{Z}$ for the canonical pairing. These data should satisfy:

- $(\alpha_i, \alpha_i) = 2$ for any $i \in I$,
- $\langle i, \lambda\rangle := \langle h_i, \lambda\rangle = (\alpha_i, \lambda)$ for $i \in I$ and $\lambda \in X$,
- $(\alpha_i, \alpha_j) \in \{0, -1\}$ for $i, j \in I$ with $i \neq j$,
- $\langle h_j, \Lambda_i\rangle = \delta_{ij}$ for all $i, j \in I$.

Hence $(a_{ij})_{i,j\in I}$ is a symmetrizable generalized Cartan matrix, where $a_{ij} = \langle h_i, \alpha_j\rangle = (\alpha_i, \alpha_j)$. We denote by $X^+ \subset X$ the dominant weights which are of the form $\sum_i \lambda_i \Lambda_i$ where $\lambda_i \geq 0$.

Associated to a symmetric Cartan data is a graph Γ without loops or multiple edges. The vertices of Γ are the elements of the set I and there is an edge from vertex i to vertex j if and only if $(\alpha_i, \alpha_j) = -1$.

The quantum group $\mathbf{U} = \mathbf{U}_q(\mathfrak{g})$ associated to a simply-laced root datum as above is the unital associative $\mathbb{Q}(q)$-algebra given by generators E_i, F_i, K_μ for $i \in I$ and $\mu \in X^\vee$, subject to the relations:

i) $K_0 = 1$, $K_\mu K_{\mu'} = K_{\mu+\mu'}$ for all $\mu, \mu' \in X^\vee$,
ii) $K_\mu E_i = q^{\langle \mu, i\rangle} E_i K_\mu$ for all $i \in I$, $\mu \in X^\vee$,
iii) $K_\mu F_i = q^{-\langle \mu, i\rangle} F_i K_\mu$ for all $i \in I$, $\mu \in X^\vee$,
iv) $E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}$, where $K_i = K_{\alpha_i}$,
v) For all $i \neq j$

$$\sum_{a+b=-\langle i,j\rangle+1} (-1)^a E_i^{(a)} E_j E_i^{(b)} = 0 \qquad \text{and} \qquad \sum_{a+b=-\langle i,j\rangle+1} (-1)^a F_i^{(a)} F_j F_i^{(b)} = 0,$$

where $E_i^{(a)} = E_i^a/[a]!$, $F_i^{(a)} = F_i^a/[a]!$, with $[a]! = \prod_{m=1}^a \frac{q^m - q^{-m}}{q - q^{-1}}$.

We are primarily interested in the idempotent form $\dot{\mathbf{U}}_q(\mathfrak{g})$ of $\mathbf{U}_q(\mathfrak{g})$. The $\mathbb{Q}(q)$-algebra $\dot{\mathbf{U}} = \dot{\mathbf{U}}_q(\mathfrak{g})$ is obtained from $\mathbf{U}$ by replacing the unit with a collection of orthogonal idempotents 1_λ for each $\lambda \in X$,

$$1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'} 1_{\lambda'}, \tag{3.1}$$

such that

$$K_\mu 1_\lambda = 1_\lambda K_\mu = q^{\langle \mu, \lambda\rangle} 1_\lambda, \quad E_i 1_\lambda = 1_{\lambda+\alpha_i} E_i, \qquad F_i 1_\lambda = 1_{\lambda-\alpha_i} F_i. \tag{3.2}$$

The algebra $\dot{\mathbf{U}}$ decomposes as direct sum of weight spaces

$$\dot{\mathbf{U}} = \bigoplus_{\lambda,\lambda'\in X} 1_{\lambda'} \dot{\mathbf{U}} 1_\lambda.$$

We say that λ, respectively λ', is the right, respectively left, weight of $x \in 1_{\lambda'} \dot{\mathbf{U}} 1_\lambda$. The algebra ${}_{\mathcal{A}}\dot{\mathbf{U}}$ is the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\mathcal{U}$ generated by products of divided powers $E_i^{(a)} 1_\lambda$ and $F_i^{(a)} 1_\lambda$, and has a similar weight decomposition

$${}_{\mathcal{A}}\dot{\mathbf{U}} = \bigoplus_{\lambda,\lambda'\in X} 1_{\lambda'} ({}_{\mathcal{A}}\dot{\mathbf{U}}) 1_\lambda.$$

3.2. **Definition of the current algebra $\mathbf{U}(\mathfrak{g}[t])$.** First, assume that $\Bbbk$ is a field of characteristic 0. The current algebra $\mathbf{U}_{\Bbbk}(\mathfrak{g}[t])$ is generated over $\Bbbk$ by $x^+_{i,r}$, $x^-_{i,s}$ and $\xi_{i,k}$ for $r,s,k \in \mathbb{N}\cup\{0\}$ and $i \in I$, modulo the following relations:

C1: For $i,j \in I$ and $r,s \in \mathbb{N}\cup\{0\}$

$$[\xi_{i,r}, \xi_{j,s}] = 0$$

C2: For $i,j \in I$ and $r \in \mathbb{N}\cup\{0\}$

$$[\xi_{i,0}, x^{\pm}_{j,r}] = \pm a_{ij} x^{\pm}_{j,r}$$

C3: For $i,j \in I$ and $r \in \mathbb{N}$, $s \in \mathbb{N}\cup\{0\}$

$$[\xi_{i,r}, x^{\pm}_{j,s}] = \pm a_{ij} x^{\pm}_{j,r+s}$$

C4: For $i,j \in I$ and $r,s \in \mathbb{N}\cup\{0\}$

$$[x^{\pm}_{i,r+1}, x^{\pm}_{j,s}] = [x^{\pm}_{i,r}, x^{\pm}_{j,s+1}]$$

C5: For $i,j \in I$ and $r,s \in \mathbb{N}\cup\{0\}$

$$[x^{+}_{i,r}, x^{-}_{j,s}] = \delta_{i,j}\xi_{i,r+s}$$

C6: Let $i \neq j$. If $a_{ij}=0$, then for $r,s \in \mathbb{N}\cup\{0\}$

$$[x^{\pm}_{i,r}, x^{\pm}_{j,s}] = 0.$$

If $a_{ij}=-1$, then for $r_1,r_2,s \in \mathbb{N}\cup\{0\}$

$$[x^{\pm}_{i,r_1}, [x^{\pm}_{i,r_2}, x^{\pm}_{j,s}]] = 0.$$

Note that $x^{\pm}_{i,s} := x^{\pm}_i \otimes t^s$, $\xi_{i,k} := \xi_i \otimes t^k$ for $i \in I$, where x^+_i, x^-_i and ξ_i are the standard generators of $\mathbf{U}(\mathfrak{g})$. We define

$$|x^{\pm}_{i,j}| = \pm\alpha_i, \quad |\xi_{j,s}| = 0\,.$$

Instead of **C3**, some authors use the relation

C3': For any $i,j \in I$ and $r,s \in \mathbb{N}\cup\{0\}$

$$[\xi_{i,r+1}, x^{\pm}_{j,s}] = [\xi_{i,r}, x^{\pm}_{j,s+1}],$$

which together with **C2** implies **C3**. The current algebra is closely connected to the Yangian, which can be thought of as its quantized universal enveloping algebra (see, for example [2]).

For a field $\Bbbk$ of characteristic p, we should use a divided power version of the current algebra. Consider the subalgebra $\mathbf{U}_{\mathbb{Z}}(\mathfrak{g}[t])$ be the subalgebra of $\mathbf{U}_{\mathbb{Q}}(\mathfrak{g}[t])$ generated over $\mathbb{Z}$ by $(x^{\pm}_{i,a})^r/r!$. For a general field $\Bbbk$, we let $\mathbf{U}_{\Bbbk}(\mathfrak{g}[t]) \cong \mathbf{U}_{\mathbb{Z}}(\mathfrak{g}[t]) \otimes_{\mathbb{Z}} \Bbbk$. We'll typically leave out the $\Bbbk$ in the notation as understood.

3.2.1. *Triangular decomposition.* Let $\mathbf{U}^+(\mathfrak{g}[t])$, $\mathbf{U}^-(\mathfrak{g}[t])$ and $\mathbf{U}^0(\mathfrak{g}[t])$ be the subalgebras of $\mathbf{U}(\mathfrak{g}[t])$ generated by $\{(x^+_{i,r})^n/n! \mid i \in I, r \in \mathbb{N}\cup\{0\}\}$, $\{(x^-_{i,r})^n/n! \mid i \in I, r \in \mathbb{N}\cup\{0\}\}$ and $\{\xi_{i,r} \mid i \in I, r \in \mathbb{N}\cup\{0\}\}$, respectively. It is well known that every element $f \in \mathbf{U}(\mathfrak{g}[t])$ can be expressed as a sum

$$f = \sum f^+ f^0 f^- \quad \text{where} \quad f^{\pm} \in \mathbf{U}^{\pm}(\mathfrak{g}[t]), f^0 \in \mathbf{U}^0(\mathfrak{g}[t]).$$

3.3. **The idempotent form.** The idempotented version $\dot{\mathbf{U}}(\mathfrak{g}[t])$ of the current algebra is a $\Bbbk$-linear category, whose objects are $\lambda \in X$. For $\lambda, \mu \in X$, the $\Bbbk$-vector space of morphisms from λ to μ is defined as follows:

$$\dot{\mathbf{U}}(\mathfrak{g}[t])(\lambda, \mu) := \mathbf{U}(\mathfrak{g}[t])/I_\xi$$

where

$$I_\xi := \sum_{i \in I} \mathbf{U}(\mathfrak{g}[t]) \left(\xi_{i,0} - \langle i, \lambda \rangle\right) + \sum_{i \in I} \left(\xi_{i,0} - \langle i, \mu \rangle\right) \mathbf{U}(\mathfrak{g}[t]) \,.$$

We will denote the identity morphism of $\lambda \in X$ in $\dot{\mathbf{U}}(\mathfrak{g}[t])(\lambda, \lambda)$ by 1_λ. The element in $\dot{\mathbf{U}}(\mathfrak{g}[t])(\lambda, \mu)$ represented by $x \in \mathbf{U}(\mathfrak{g}[t])$ can be written as $1_\mu x 1_\lambda = 1_\mu x = x 1_\lambda$, $\mu - \lambda = |x|$. The composition in $\dot{\mathbf{U}}(\mathfrak{g}[t])$ is induced by multiplication in the current algebra, i.e.

$$(1_\mu x 1_\nu)(1_\nu y 1_\lambda) = 1_\mu x y 1_\lambda$$

for $x, y \in \mathbf{U}(\mathfrak{g}[t])$, $\lambda, \mu, \nu \in X$, which is zero unless $|x| = \mu - \nu$, $|y| = \nu - \lambda$.

3.3.1. *Triangular decomposition.* Let $\dot{\mathbf{U}}^+(\mathfrak{g}[t])$ and $\dot{\mathbf{U}}^-(\mathfrak{g}[t])$ be linear subcategories of $\dot{\mathbf{U}}(\mathfrak{g}[t])$ whose hom spaces between λ and μ are

$$1_\mu \mathbf{U}^+(\mathfrak{g}[t]) 1_\lambda := \{1_\mu x^+ 1_\lambda \mid x^+ \in \mathbf{U}^+(\mathfrak{g}[t])\}$$

and

$$1_\mu \mathbf{U}^-(\mathfrak{g}[t]) 1_\lambda := \{1_\mu x^- 1_\lambda \mid x^- \in \mathbf{U}^-(\mathfrak{g}[t])\} \,,$$

respectively. Let

$$\dot{\mathbf{U}}^0(\mathfrak{g}[t]) := \oplus_\lambda 1_\lambda \mathbf{U}^0(\mathfrak{g}[t]) 1_\lambda$$

be the center (of objects) of $\dot{\mathbf{U}}(\mathfrak{g}[t])$. Then any morphism f of $\dot{\mathbf{U}}(\mathfrak{g}[t])$ decomposes as

$$f = \sum f^+ f^0 f^- \quad \text{where} \quad f^\pm \in \dot{\mathbf{U}}^\pm(\mathfrak{g}[t]), f^0 \in \dot{\mathbf{U}}^0(\mathfrak{g}[t]).$$

3.3.2. *Grading.* Both $\dot{\mathbf{U}}(\mathfrak{g}[t])$ and $\mathbf{U}(\mathfrak{g}[t])$ are naturally graded. We'll take the convention that for $X \in \mathfrak{g}$, we have that $X \otimes t^m$ has degree $2m$.

3.3.3. *Shifting.* For each $\xi \in \Bbbk$, the loop algebra is equipped with an automorphism $\tau_\xi(X \otimes t^m) = X \otimes (t - \xi)^m$ for any $X \in \mathfrak{g}$. For any module V over $\mathbf{U}(\mathfrak{g}[t])$, we can precompose its action with this automorphism to obtain a new module V_ξ, which we will call the **shift** of V by ξ.

3.3.4. *Weyl modules.* For a fixed λ, let $m_i = \langle i, \lambda \rangle$. Recall that the universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ has a finite dimensional representation called the **(finite) Weyl module** $V(\lambda)$. We add the word "finite" here to avoid any confusion with the corresponding modules over the current algebra. These are modules generated $\mathbf{U}(\mathfrak{g})$ by a single vector v_λ with defining relations:

$$\mathfrak{g}^+ v_\lambda = 0, \quad \xi_i v_\lambda = \langle i, \lambda \rangle v_\lambda, \quad (x_i^-)^{(m_i+1)} v_\lambda = 0 \qquad \text{for any } i \in I. \tag{3.3}$$

If $\Bbbk$ has characteristic 0, then these modules give a complete, irredundant list of the finite dimensional simple modules over $\mathbf{U}(\mathfrak{g})$. If $\Bbbk$ has positive characteristic, then for most $\langle$ these will have proper submodules, and the finite dimensional simple modules are their unique simple quotients.

Now, we discuss analogs of these modules over the current algebra. The **global Weyl module** $\mathbb{W}(\lambda)$ is the $\mathfrak{g}[t]$-module generated over $\mathbf{U}(\mathfrak{g}[t])$ by an element w_λ with defining relations:

$$\mathfrak{g}^+[t] w_\lambda = 0, \quad \xi_{i,0} w_\lambda = \langle i, \lambda \rangle w_\lambda, \quad (x_{i,0}^-)^{m_i+1} w_\lambda = 0 \qquad \text{for any } i \in I. \tag{3.4}$$

The ring $\mathbf{U}(\mathfrak{h}[\mathbf{t}])$ (which can be thought of as a polynomial ring in infinitely many variables) has a right action on $\mathbb{W}(\lambda)$ sending $u w_\lambda \cdot h = u h w_\lambda$. This action is not faithful, but rather factors through a finitely generated quotient $\mathbb{A}_\lambda$. By [14, 6.1], the ring $\mathbb{A}_\lambda$ is a polynomial ring generated by an alphabet $\{\mathsf{x}_{i,1}, \ldots, \mathsf{x}_{i,\langle i,\lambda \rangle}\}_{i \in I}$ with $\mathsf{x}_{i,k}$ having degree $2k$. In particular, its Hilbert series is

$\prod_{i\in I}(1-t)^{-1}\cdots(1-t^{\langle i,\lambda\rangle})^{-1}$. Note that a maximal ideal in $\mathbb{A}_\lambda$ is naturally encoded by scalars $\nu_{i,k}$ given by the image of $\mathsf{x}_{i,k}$; we'll usually consider these as polynomials

$$\boldsymbol{\nu}_i(-z) = z^{\langle i,\lambda\rangle} + \nu_{i,1}z^{\langle i,\lambda\rangle-1} + \cdots + \nu_{i,\langle i,\lambda\rangle}$$

For a Lie algebra $\mathfrak{a}$, let us denote by $\mathfrak{a}t[t]$ the ideal of $\mathfrak{a}[t]$ generated by the elements of the form $x\otimes t^n$ with $x\in\mathfrak{a}$ and $n>0$.

The **local Weyl module** $W(\lambda)$ is the $\mathfrak{g}[t]$-module generated by an element w_λ with defining relations:

$$\mathfrak{g}^+[t]w_\lambda = 0,\quad \mathfrak{h}t[t]w_\lambda = 0,\quad \xi_{i,0}w_\lambda = \langle i,\lambda\rangle w_\lambda,\quad (x_{i,0}^-)^{m_i+1}w_\lambda = 0 \qquad \text{for any } i\in I. \tag{3.5}$$

We can also consider the shifts of these modules by scalars $W_\xi(\lambda), \mathbb{W}_\xi(\lambda)$; we'll call these **shifted Weyl modules**. These arise naturally in the structure theory of these modules, since:

Lemma 3.1 ([14, 5.8])**.** The specialization of $\mathbb{W}(\lambda)$ at the maximal ideal for $\boldsymbol{\nu}_i$ in $\mathbb{A}_\lambda$ is isomorphic (after possible finite base extension) to the tensor product $\bigotimes_\xi W_\xi(\lambda_\xi)$ where ξ ranges over the roots $\boldsymbol{\nu}_i(\xi)=0$ for all i, and λ_ξ are roots such that $\langle i,\lambda_\xi\rangle$ is the multiplicity of ξ as a root of $\boldsymbol{\nu}_i(z)$, and $\sum_\xi \lambda_\xi = \lambda$.

As long as $\mathfrak{g}$ is finite type, this decomposition is unique: λ_ξ is the sum of the fundamental weights with coefficients given by ξ's multiplicities as roots of $\boldsymbol{\nu}_i(z)$. For infinite type, this decomposition is not unique, since there exist weights orthogonal to all simple coroots h_i.

Both of these modules are naturally graded with the generating vector having degree 0, since the relations (3.4) and (3.5) are homogeneous.

The global (resp. local) Weyl modules have a natural universal property: there is a homomorphism of $W(\lambda)$ (resp. $\mathbb{W}(\lambda)$) to an integrable module M sending $w_\lambda\to m\in M$ if and only if $\mathfrak{g}^+[t]m=0$ and $\xi_{i,0}m=\langle i,\lambda\rangle m$ (resp. also $\mathfrak{h}t[t]m=0$). In particular, this map will be surjective if m generates M, and homogeneous of degree k if M is graded with m of degree k.

3.3.5. *Evaluation modules.* For every $\chi\in A$ in some $\Bbbk$-algebra A, we have an evaluation homomorphism $\mathrm{ev}_\chi\colon \mathfrak{g}[t]\to\mathfrak{g}\otimes A$ sending $x\otimes t^i\mapsto \chi^i x$ for $x\in\mathfrak{g}$. For any representation V of $\mathfrak{g}\otimes A$, we have an induced pullback representation V_χ over $\mathfrak{g}[t]$. Particularly interesting cases include:

- $A=\Bbbk$. In this case, if V is an irrep over $\mathfrak{g}$, then V_χ will also be irreducible.
- $A=\Bbbk[x]$. In this case, we have the universal evaluation module V_x.

Note that the shift by $\xi\in\Bbbk$ of an evaluation module for $\chi\in A$ is again an evaluation module with parameter $\chi+\xi$. Thus, our notations for shift and evaluation will not conflict.

Note that since the highest weight vector in $V_\chi(\lambda)$ satisfies the equations (3.4), we have a surjective map $\mathbb{W}(\lambda)\to V_\chi(\lambda)$. More generally, assume that $\chi_1,\dots,\chi_N$ are distinct scalars.

Lemma 3.2. We have a surjective map

$$\mathbb{W}(\lambda_1+\cdots+\lambda_N)\to V_{\chi_1}(\lambda_1)\otimes\cdots\otimes V_{\chi_N}(\lambda_N),$$

sending

$$w_{\lambda_1+\cdots+\lambda_N}\mapsto v_{\lambda_1}\otimes\cdots\otimes v_{\lambda_N}.$$

Proof. The existence of this map is clear from the universal property. To show that this map is surjective as well, note that by Lagrange interpolation, there exists a polynomial f_i such that $f_i(\chi_j)=\delta_{ij}$. In this case, $X\otimes f(t)$ acts on $V_{\chi_1}(\lambda_1)\otimes\cdots\otimes V_{\chi_N}(\lambda_N)$ by $1\otimes\cdots\otimes X\otimes\cdots\otimes 1$, that is, by X in the ith tensor factor. Since the tensor product of highest weight vectors generates under the action of these operators, we're done. $\square$

Note that the map factors through $W_{\chi_1}(\lambda_1)\otimes\cdots\otimes W_{\chi_N}(\lambda_N)$.

Lemma 3.3. For any finite collection of linearly independent elements $u_i \in \mathbf{U}(\mathfrak{g}[\mathbf{t}])$, there is an integer N such that for generic $\chi_1, \dots, \chi_N$ the images $z_i = (\mathrm{ev}_{\chi_1} \otimes \cdots \otimes \mathrm{ev}_{\chi_N}) \circ \Delta^{(N)}(u_i)$ under the N-fold coproduct with the universal evaluation in N independent parameters remain linearly independent.

Proof. To show this result for generic $\chi_1, \dots, \chi_N$ is the same as to show it for the universal evalution over $\mathcal{K}[x_1, \dots, x_n]$. We can assume that each u_i is a PBW monomial without loss of generality. In this case, we let N be the maximal length of one of these monomials. That is, we consider $u_i = (X_{a_1} \otimes t^{b_1})(X_{a_2} \otimes t^{b_2}) \cdots (X_{a_n} \otimes t^{b_n})$ where $\{X_1, \dots, X_d\}$ is a basis of $\mathfrak{g}$. The N-fold coproduct $\Delta^{(N)}(u_i)$ is of the form

$$(X_{a_1} \otimes t^{b_1}) \otimes (X_{a_2} \otimes t^{b_2}) \otimes \cdots \otimes (X_{a_n} \otimes t^{b_n}) \otimes 1 \otimes \cdots \otimes 1 + \cdots .$$

Note that this term does not appear in the coproduct of any other PBW monomial.

Now, applying the evaluation, we have the form

$$z_i = x_1^{b_1} \cdots x_n^{b_n} X_{a_1} \otimes X_{a_2} \otimes \cdots \otimes X_{a_n} \otimes 1 \otimes \cdots \otimes 1 + \cdots$$

Again, this term will not show up in any other PBW monomial. This shows the desired independence. □

The following lemma is presumably standard, but we include a short proof for completeness.

Lemma 3.4. For any element $u \in \mathbf{U}(\mathfrak{g})$, there is a tensor product of two Weyl modules for $\mathfrak{g}$ on which it acts non-trivially.

Proof. This is a straightforward consequence of Peter-Weyl if $\mathcal{K}$ has characteristic 0. However, let us give an argument that works in arbitrary characteristic. By PBW, we can write $u = \sum u_i^0 u_i^+ u_i^-$ where $u_i^\pm \in \mathbf{U}^\pm(\mathfrak{g})$. Now, consider the action of u on the tensor product of the highest and lowest weight vectors $v^+ \otimes v^- \in V(\lambda_+) \otimes V(\lambda_-)$ for some λ_+ and λ_-. We can assume without loss of generality that these elements are weight vectors, and that we have used a minimal number of terms subject to this restriction.

Since all elements of $\mathbf{U}^\pm(\mathfrak{g})$ kill $v_\pm$, we have that

$$u(v^+ \otimes v^-) = \sum u_i^0 u_i^+ u_i^- (v^+ \otimes v^-) = \sum u_i^0 (u_i^- v^+ \otimes u_i^+ v^- + \cdots)$$

where the remaining terms have higher weight in the left term and lower in the right term.

For any linearly independent subset $\{w_i^\pm\}$ of $\mathbf{U}^\pm(\mathfrak{g})$, the set $\{w_i^\pm v^\pm\}$ is linearly independent for $\lambda_\pm \gg 0$. Thus, for $\lambda_\pm \gg 0$, the terms of minimal weight in the left term and maximal in the right term give a linear combination $\sum u_i^0 (u_i^+ v^+ \otimes u_i^- v^-) = 0$. Since these are weight vectors, we have obtained a linear dependence in the set $\{u_i^+ u_i^-\}$; we can use this to reduce the number of terms in the sum of u, obtaining a contradiction to the assumption that we had taken the minimal number possible. □

Lemma 3.5. No element of $\mathbf{U}(\mathfrak{g}[t])$ kills all global Weyl modules.

Proof. We must show that no $u \in \mathbf{U}(\mathfrak{g}[t])$ can act trivially on all global Weyl modules. By Lemma 3.3, for generic χ_i, we have $v = (\mathrm{ev}_{\chi_1} \otimes \cdots \otimes \mathrm{ev}_{\chi_N}) \circ \Delta^{(N)}(u) \neq 0$. Such a set of χ_i must exist if $\Bbbk$ is infinite (and we can replace $\Bbbk$ with an infinite extension without changing the result).

Thus, we have an algebra map $\mathbf{U}(\mathfrak{g}[t]) \to \mathbf{U}(\mathfrak{g})^{\otimes N} \cong \mathbf{U}(\mathfrak{g}^{\oplus N})$ which does not kill u. Applying Lemma 3.4 to $\mathbf{U}(\mathfrak{g}^{\oplus N})$, we have a tensor product

$$(V(\lambda_{1,1}) \otimes \cdots \otimes V(\lambda_{1,k_1})) \boxtimes \cdots \boxtimes (V(\lambda_{N,1}) \otimes \cdots \otimes V(\lambda_{N,k_N}))$$

on which v acts non-trivially. Here the symbol $\boxtimes$ is used for the tensor product of $\mathfrak{g}[t]$-modules, where the standard symbol $\otimes$ refers to the tensor products of $\mathfrak{g}$-modules.

That is to say, u acts non-trivially on

$$(V_{\chi_1}(\lambda_{1,1}) \otimes \cdots \otimes V_{\chi_1}(\lambda_{1,k_1})) \otimes \cdots \otimes (V_{\chi_N}(\lambda_{N,1}) \otimes \cdots \otimes V_{\chi_N}(\lambda_{N,k_N}).$$

Thus, necessarily, this shows that u acts non-trivially on $V_{x_{1,1}}(\lambda_{1,1}) \otimes \cdots \otimes V_{x_{N,k_N}}(\lambda_{N,k_N})$, for formal parameters $x_{*,*}$, and thus also when $x_{i,k}$ is replaced by a generic numerical parameter $\chi_{i,k} \in \Bbbk$.

Thus u must act non-trivially on $V_{\chi_{1,1}}(\lambda_{1,1}) \otimes \cdots \otimes V_{\chi_{N,k_N}}(\lambda_{N,k_N})$. Since this is a quotient of the global Weyl module $\mathbb{W}(\lambda_{1,1} + \cdots + \lambda_{N,k_N})$, the action on this module must be non-trivial as well. □

4. Categorified quantum groups

Here we describe a categorification of $\mathbf{U}(\mathfrak{g})$ mainly following [12] and [25]. For an elementary introduction to the categorification of $\mathfrak{sl}_2$ see [28].

4.0.6. *Choice of scalars Q.* Let $\Bbbk$ be an field, not necessarily algebraically closed, or characteristic zero.

Definition 4.1. Associated to a symmetric Cartan datum define an *choice of scalars Q* consisting of:

- $\{t_{ij} \mid \text{ for all } i,j \in I\}$,

such that

- $t_{ii} = 1$ for all $i \in I$ and $t_{ij} \in \Bbbk^{\times}$ for $i \neq j$,
- $t_{ij} = t_{ji}$ when $a_{ij} = 0$.

We say that a choice of scalars Q is *integral* if $t_{ij} = \pm 1$ for all $i, j \in I$.

The relevant parameters that govern the behavior of the 2-category $\mathcal{U}_Q(\mathfrak{g})$ are the products $v_{ij} = t_{ij}^{-1} t_{ji}$ taken over all pairs $i, j \in I$. It is possible to define $\mathcal{U}_Q(\mathfrak{g})$ over the integers, rather than a base field $\Bbbk$, and for this reason, an integral choice of scalars is the most natural.

4.1. Definition of the 2-category $\mathcal{U}_Q(\mathfrak{g})$.

By a graded linear 2-category we mean a category enriched in graded linear categories, so that the hom spaces form graded linear categories, and the composition map is grading preserving.

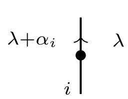
Given a fixed choice of scalars Q we can define the following 2-category.

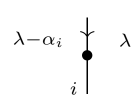
Definition 4.2. The 2-category $\mathcal{U}_Q(\mathfrak{g})$ is the graded linear 2-category consisting of:

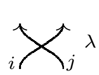
- objects λ for $\lambda \in X$.
- 1-morphisms are formal direct sums of (shifts of) compositions of

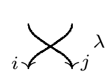
$$\mathbf{1}_\lambda, \quad \mathbf{1}_{\lambda+\alpha_i}\mathcal{E}_i = \mathbf{1}_{\lambda+\alpha_i}\mathcal{E}_i\mathbf{1}_\lambda = \mathcal{E}_i\mathbf{1}_\lambda, \quad \text{and} \quad \mathbf{1}_{\lambda-\alpha_i}\mathcal{F}_i = \mathbf{1}_{\lambda-\alpha_i}\mathcal{F}_i\mathbf{1}_\lambda = \mathcal{F}_i\mathbf{1}_\lambda$$

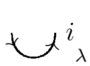
  for $i \in I$ and $\lambda \in X$. In particular, any morphism can be written as a finite formal sum of symbols $\mathcal{E}_{\boldsymbol{i}}\mathbf{1}_\lambda\langle t\rangle$ where $\boldsymbol{i} = (\pm i_1, \ldots, \pm i_m)$ is a signed sequence of simple roots, t is a grading shift, $\mathcal{E}_{+i}\mathbf{1}_\lambda := \mathcal{E}_i\mathbf{1}_\lambda$ and $\mathcal{E}_{-i}\mathbf{1}_\lambda := \mathcal{F}_i\mathbf{1}_\lambda$, and $\mathcal{E}_{\boldsymbol{i}}\mathbf{1}_\lambda\langle t\rangle := \mathcal{E}_{\pm i_1}\ldots\mathcal{E}_{\pm i_m}\mathbf{1}_\lambda\langle t\rangle$.
- 2-morphisms are $\Bbbk$-vector spaces spanned by compositions of (decorated) tangle-like diagrams illustrated below.

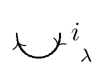
 $: \mathcal{E}_i\mathbf{1}_\lambda \to \mathcal{E}_i\mathbf{1}_\lambda\langle(\alpha_i,\alpha_i)\rangle$

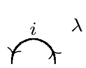
 $: \mathcal{F}_i\mathbf{1}_\lambda \to \mathcal{F}_i\mathbf{1}_\lambda\langle(\alpha_i,\alpha_i)\rangle$

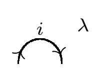
 $: \mathcal{E}_i\mathcal{E}_j\mathbf{1}_\lambda \to \mathcal{E}_j\mathcal{E}_i\mathbf{1}_\lambda\langle-(\alpha_i,\alpha_j)\rangle$

 $: \mathcal{F}_i\mathcal{F}_j\mathbf{1}_\lambda \to \mathcal{F}_j\mathcal{F}_i\mathbf{1}_\lambda\langle-(\alpha_i,\alpha_j)\rangle$

 $: \mathbf{1}_\lambda \to \mathcal{F}_i\mathcal{E}_i\mathbf{1}_\lambda\langle 1+(\lambda,\alpha_i)\rangle$

 $: \mathbf{1}_\lambda \to \mathcal{E}_i\mathcal{F}_i\mathbf{1}_\lambda\langle 1-(\lambda,\alpha_i)\rangle$

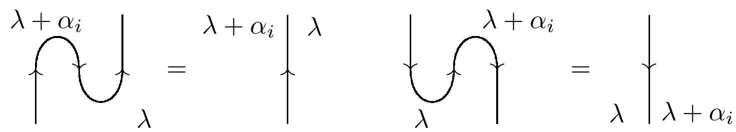
 $: \mathcal{F}_i\mathcal{E}_i\mathbf{1}_\lambda \to \mathbf{1}_\lambda\langle 1+(\lambda,\alpha_i)\rangle$

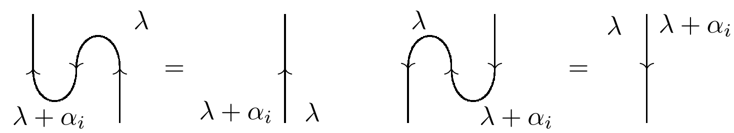
 $: \mathcal{E}_i\mathcal{F}_i\mathbf{1}_\lambda \to \mathbf{1}_\lambda\langle 1-(\lambda,\alpha_i)\rangle$

Here we follow the grading conventions in [12] and [29] which are opposite to those from [25]. In this 2-category (and those throughout the paper) we read diagrams from right to left and bottom to top. The identity 2-morphism of the 1-morphism $\mathcal{E}_i\mathbf{1}_\lambda$ is represented by an upward oriented line labeled by i and the identity 2-morphism of $\mathcal{F}_i\mathbf{1}_\lambda$ is represented by a downward such line.

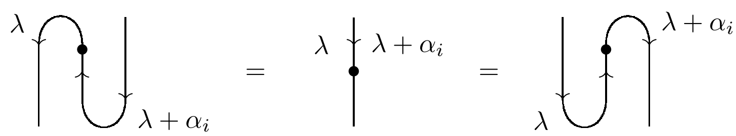
The 2-morphisms satisfy the following relations:

(1) The 1-morphisms $\mathcal{E}_i\mathbf{1}_\lambda$ and $\mathcal{F}_i\mathbf{1}_\lambda$ are biadjoint (up to a specified degree shift). These conditions are expressed diagrammatically as

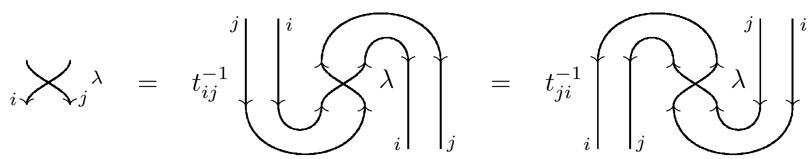
(4.1) 

(4.2) 

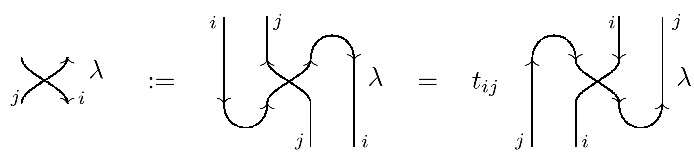
(2) The 2-morphisms are Q-cyclic with respect to this biadjoint structure.

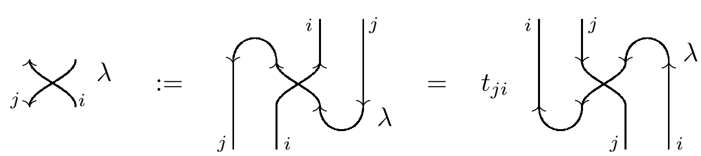
(4.3) 

The Q-cyclic relations for crossings are given by

(4.4) 

Sideways crossings can then be defined utilizing the Q-cyclic condition by the equalities:

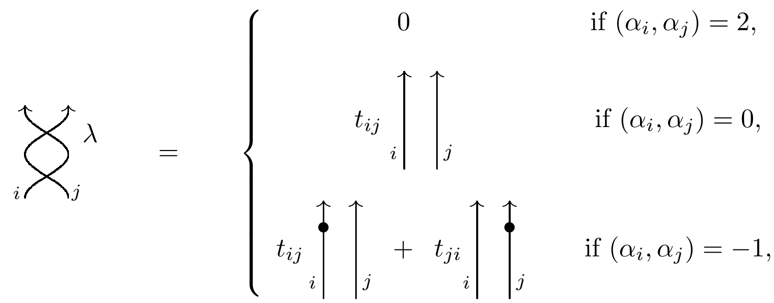
(4.5) 

(4.6) 

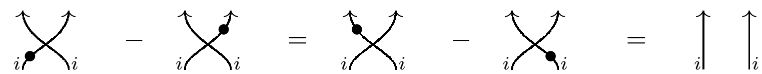
where the second equality in (4.5) and (4.6) follow from (4.4).

(3) The $\mathcal{E}$'s carry an action of the KLR algebra associated to Q. The KLR algebra $R = R_Q$ associated to Q is defined by finite $\Bbbk$-linear combinations of braid–like diagrams in the plane, where each strand is labeled by a vertex $i \in I$. Strands can intersect and can carry dots but triple intersections are not allowed. Diagrams are considered up to planar isotopy that do not change the combinatorial type of the diagram. We recall the local relations:

i) For $i \neq j$

(4.7)


ii) The nilHecke dot sliding relations

(4.8)


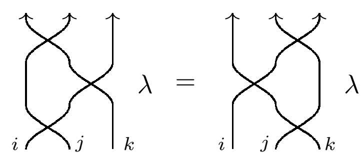
hold.

iii) For $i \neq j$ the dot sliding relations

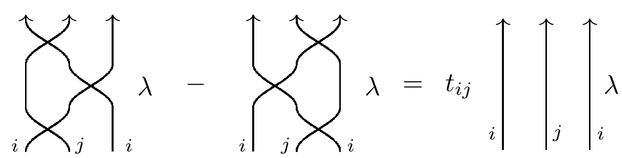
(4.9)


hold.

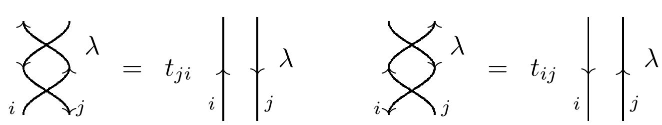
iv) Unless $i = k$ and $(\alpha_i, \alpha_j) < 0$ the relation

(4.10)


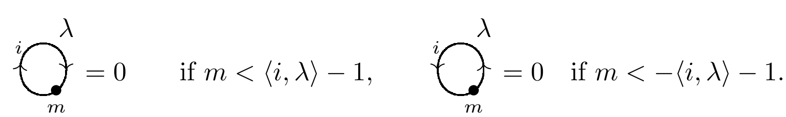
holds. Otherwise, $(\alpha_i, \alpha_j) = -1$ and

(4.11)


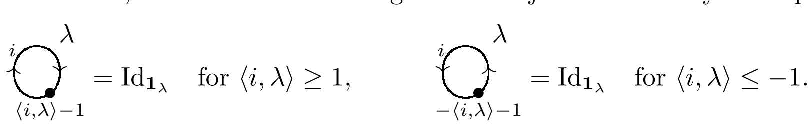
(4) When $i \neq j$ one has the mixed relations relating $\mathcal{E}_i\mathcal{F}_j$ and $\mathcal{F}_j\mathcal{E}_i$:

(4.12)


(5) Negative degree bubbles are zero. That is, for all $m \in \mathbb{Z}_+$ one has

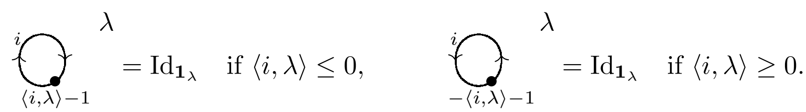
(4.13)


On the other hand, a dotted bubble of degree zero is just the identity 2-morphism:



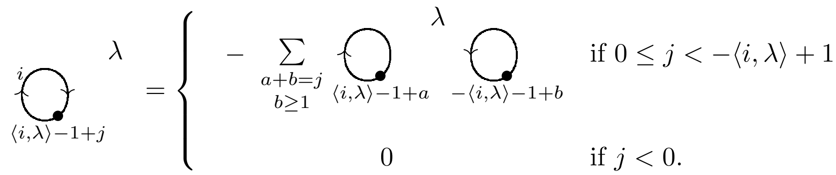
(6) For any $i \in I$ one has the extended $\mathfrak{sl}_2$-relations. In order to describe certain extended $\mathfrak{sl}_2$ relations it is convenient to use a shorthand notation from [27] called fake bubbles. These are diagrams for dotted bubbles where the labels of the number of dots is negative, but the total degree of the dotted bubble taken with these negative dots is still positive. They allow us to write these extended $\mathfrak{sl}_2$ relations more uniformly (i.e. independent on whether the weight $\langle i, \lambda\rangle$ is positive or negative).

- Degree zero fake bubbles are equal to the identity 2-morphisms



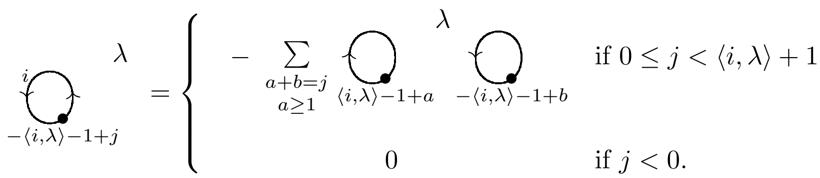
- Higher degree fake bubbles for $\langle i,\lambda\rangle < 0$ are defined inductively as

(4.14)

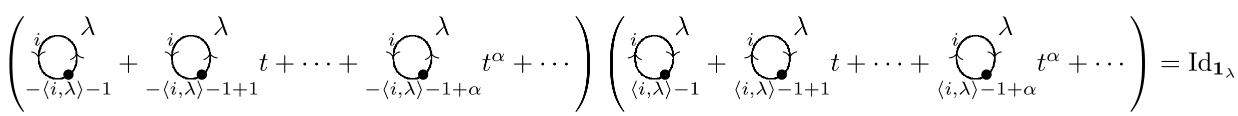


- Higher degree fake bubbles for $\langle i,\lambda\rangle > 0$ are defined inductively as

(4.15)



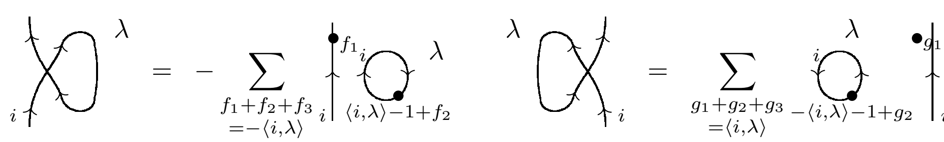
These equations arise from the homogeneous terms in t of the 'infinite Grassmannian' equation



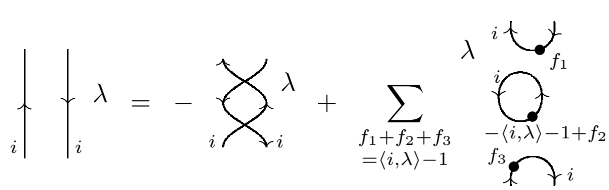
(4.16)

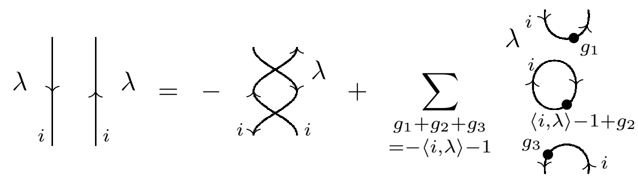
Now we can define the extended $\mathfrak{sl}_2$ relations. Note that in [12] additional curl relations were provided that can be derived from those above. For the following relations we employ the convention that all summations are increasing, so that $\sum_{f=0}^{\alpha}$ is zero if $\alpha < 0$.

(4.17)



(4.18)



(4.19)


The significance of the 2-category $\mathcal{U}_Q(\mathfrak{g})$ is given by the following theorem.

Theorem 4.3. ([25, 45]) There is an isomorphism

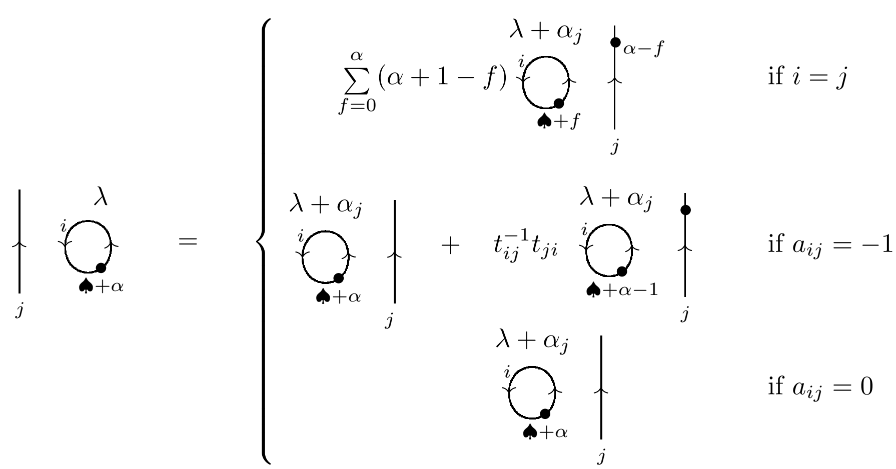
$$\gamma\colon {}_{\mathcal{A}}\dot{\mathbf{U}} \longrightarrow K_0(\dot{\mathcal{U}}) \tag{4.20}$$

of linear categories.

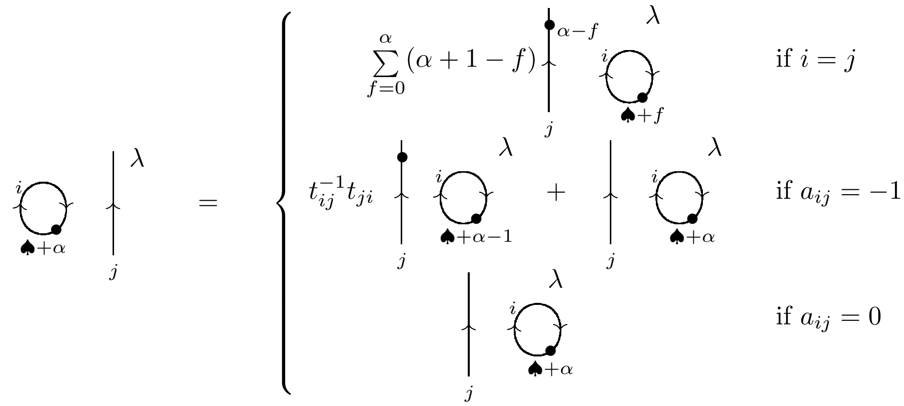
4.2. **Symmetric functions and bubbles.** The calculus of closed diagrams in the 2-category $\mathcal{U}_Q(\mathfrak{g})$ is remarkably rich. A prominent role is played by the non-nested dotted bubbles of a fixed orientation since any closed diagram in the graphical calculus for $\mathcal{U}_Q(\mathfrak{g})$ can be reduced to composites of such diagrams. In what follows it is often convenient to introduce a shorthand notation

for all $\langle i,\lambda\rangle$. Note that as long as $\alpha \geq 0$ this notation makes sense even when $\spadesuit+\alpha < 0$. These negative values are the fake bubbles defined in the previous section.

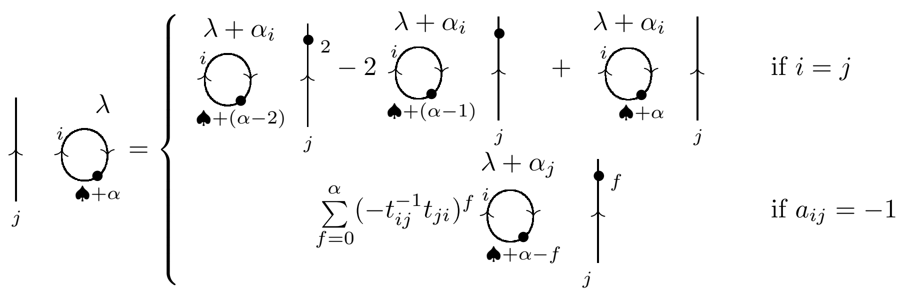
Using equations (6.8) and (6.9) of [12] one can prove that the following bubble slide equations

(4.21)


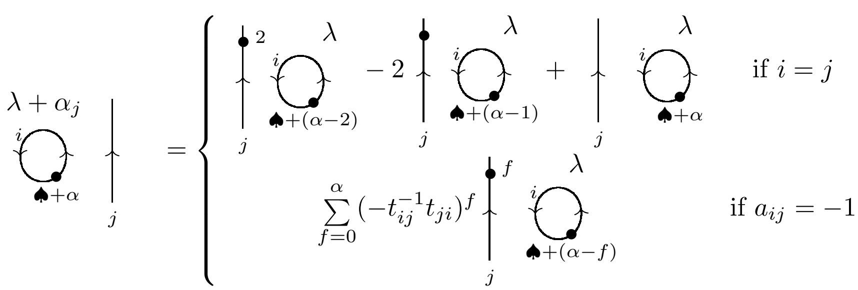
(4.22)



(4.23)



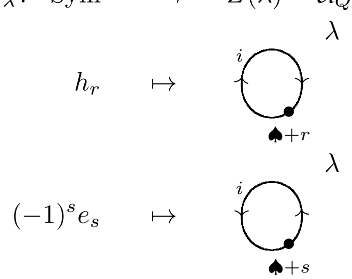
(4.24)



hold in $\mathcal{U}_Q(\mathfrak{g})$.

In [27] it is shown that there is an isomorphism

(4.25)
$$\psi_\lambda\colon\ \mathrm{Sym} \longrightarrow Z(\lambda) = \mathcal{U}_Q(\mathfrak{sl}_2)(\mathbf{1}_\lambda, \mathbf{1}_\lambda)$$



where Sym denotes the ring of symmetric functions, h_r denotes the complete symmetric function of degree r, and e_s denotes the elementary symmetric function of degree s. In fact, under this isomorphism the well known relationship between complete and elementary symmetric functions becomes the infinite Grassmannian equation (4.16).

It is well known that for a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, products of elementary symmetric functions $e_\lambda = e_{\lambda_1} \dots e_{\lambda_n}$ form a $\mathbb{Z}$-basis for Sym, see for example [30]. Likewise, products of complete symmetric functions also provide a $\mathbb{Z}$-basis for Sym. This mirrors the fact that any closed diagram in the graphical calculus for $\mathcal{U}_Q(\mathfrak{sl}_2)$ can be reduced to a product on non-nested bubbles of a given orientation.

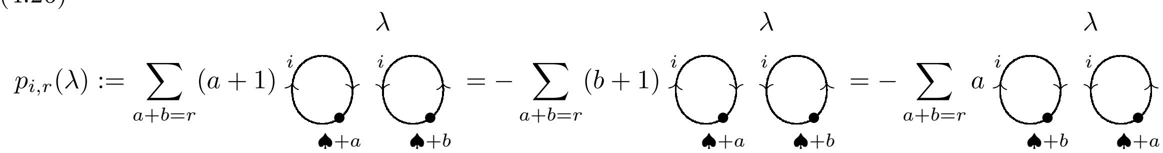
In the calculus of the 2-category $\mathcal{U}_Q(\mathfrak{g})$, we have the isomorphism

$$\psi_\lambda \colon \prod_{i \in I} \mathrm{Sym} \longrightarrow Z(\lambda) = \mathcal{U}_Q(\mathfrak{g})(\mathbf{1}_\lambda, \mathbf{1}_\lambda)$$

since any closed diagram can still be reduced to products of non-nested closed bubbles labelled by $i \in I$.

In what follows, it will be interesting to consider which products of closed diagrams correspond to the $\mathbb{Q}$-basis of Sym given by the power sum p_r symmetric functions (see e.g. p.16 in [30]). Using a formula that expresses power sum symmetric functions in terms of products of complete and elementary symmetric functions, we can denote by $p_{i,r}(\lambda)$ for $r > 0$, the image of the power sum symmetric polynomial on i-labelled strands in $Z(\lambda)$:

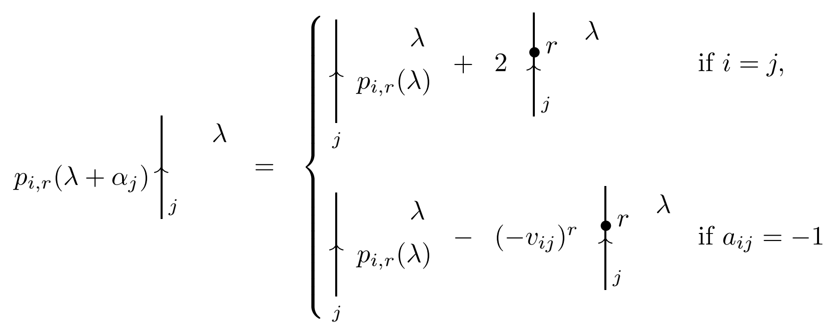
(4.26)



For later convenience we set $p_{i,0}(\lambda) = \langle i, \lambda \rangle$.

The bubble sliding equations imply the following power sum slide rule

(4.27)



4.3. **Dependence of $\mathcal{U}_Q(\mathfrak{g})$ on choice of scalars.**

The purpose of this section is to exclude some choices of scalars for our study of trace decategorifications of the 2-category $\mathcal{U}_Q(\mathfrak{g})$.

For any pair of vertices $i, j \in I$, rescaling one of the ij-crossings by $\lambda \in \Bbbk^\times$ and leaving the other one fixed has the effect of replacing scalars t_{ij} and t_{ji} with λt_{ij} and λt_{ji}. Then the 2-category resulting from this choice of scalars is isomorphic to the 2-category $\mathcal{U}_Q(\mathfrak{g})$ by the 2-functor that rescales one of the ij-crossings. Hence, the 2-category $\mathcal{U}_Q(\mathfrak{g})$ depends only on the parameters $v_{ij} := t_{ij}^{-1} t_{ji}$ taken over all pairs of vertices $i, j \in I$. Note that if $(\alpha_i, \alpha_j) = 0$, then $v_{ij} = 1$ since $t_{ij} = t_{ji}$. For an integral choice of scalars $v_{ij} = \pm 1$ for all $i, j \in I$.

The most common choices of scalars Q for the 2-category $\mathcal{U}_Q(\mathfrak{g})$ involve taking all $v_{ij} = +1$ (the unsigned and cyclic version [25]), or all $v_{ij} = -1$ (the signed, and more geometrically motivated definition [42]).

Definition 4.4. A symmetric Cartan data represented by the graph Γ is said to be colored by the elements of the set $\mathbb{Z}_2$ if it is equipped with a map $c \colon I \to \mathbb{Z}_2$. Given a coloring $c \colon I \to \mathbb{Z}_2$ of a Cartan data, a choice of scalars Q is said to be **$(+)$-compatible with the coloring** if

- $v_{ij} = +1$ if $(\alpha_i, \alpha_j) = -1$ and $c(i) = c(j)$,
- $v_{ij} = -1$ if $(\alpha_i, \alpha_j) = -1$ and $c(i) \neq c(j)$.

A choice of scalars Q is said to be $(-)$**-compatible with the coloring** if

- $v_{ij} = -1$ if $(\alpha_i, \alpha_j) = -1$ and $c(i) = c(j)$,
- $v_{ij} = +1$ if $(\alpha_i, \alpha_j) = -1$ and $c(i) \neq c(j)$.

In other words, a choice of scalars is $(\pm)$-compatible with the coloring when like colored vertices connected by an edge have $v_{ij} = \pm 1$, and oppositely colored vertices connected by an edge have $v_{ij} = \mp 1$.

Let Γ be a symmetric Cartan data equipped with a coloring $c\colon I \to \mathbb{Z}_2$. Fix an arbitrary orientation for each edge in Γ. Associated to Γ and the chosen orientation define an integral choice of scalars $(-)$-compatible with the coloring as follows:

- $t_{ij} = 1$ if $(\alpha_i, \alpha_j) = 0$,
- $t_{ij} = t_{ji} = +1$ if $(\alpha_i, \alpha_j) = -1$ and $c(i) \neq c(j)$,
- $t_{ij} = -1$ if $c(i) = c(j)$ and $i \rightarrow j$, and
- $t_{ij} = +1$ if $c(i) = c(j)$ and $i \leftarrow j$.

In other words, $v_{ij} = -1$ if the vertices have the same color, and $v_{ij} = +1$ if the vertices have different colors. Reversing the orientation of an edge between two vertices with the same color has the effect of rescaling these t_{ij} by the scalar $\lambda = -1$. Indeed, rescaling one of the ij-crossings by $\lambda = -1$ for any ij-pair allows us to replace t_{ij} by $-t_{ij}$ without affecting the parameters v_{ij}. An integral choice of scalars $(+)$-compatible with the coloring can be defined in a similar fashion.

Remark 4.5. If the graph Γ associated to the Cartan data contains no odd cycles, then we can obtain all integral choices of scalars Q (up to rescaling) from the $(-)$-compatible choice obtained from some coloring of the graph. However, if Γ does contain an odd cycle we cannot obtain the choice of scalars where $v_{ij} = +1$ for all edges of an odd cycle, since this would require the vertices of the odd cycle to alternate in color, which is impossible.

4.4. **The 2-category $\mathcal{U}_Q^*(\mathfrak{g})$.** The 2-category $\mathcal{U}^* := \mathcal{U}_Q^*(\mathfrak{g})$ is defined as follows. The objects and 1-morphisms are the same as those of $\mathcal{U}_Q(\mathfrak{g})$. Given a pair of 1-morphisms $f, g\colon n \to m$, the abelian group $\mathcal{U}^*(n,m)(f,g)$ is defined by

$$\mathcal{U}^*(n,m)(f,g) := \bigoplus_{t \in \mathbb{Z}} \mathcal{U}(n,m)(f, g\langle t \rangle).$$

The category $\mathcal{U}^*(n,m)$ is additive and enriched over $\mathbb{Z}$-graded abelian groups. Alternatively, the linear category $\mathcal{U}^*(n,m)$ is obtained from $\mathcal{U}(n,m)$ by adding a family of natural isomorphisms $f \to f\langle 1 \rangle$ for each object f of the category $\mathcal{U}(n,m)$.

In $\mathcal{U}^*(n,m)$ an object f and its translation $f\langle t \rangle$ are isomorphic via the 2-isomorphism

$$1_f \in \mathcal{U}(n,m)(f, f\langle 0 \rangle) = \mathcal{U}(n,m)(f, (f\langle t \rangle)\langle -t \rangle) \subset \mathcal{U}^*(n,m)(f, f\langle t \rangle).$$

The inverse of the isomorphism $1_f\colon f \to f\langle t \rangle$ is given by

$$1_f\langle t \rangle \in \mathcal{U}(n,m)(f\langle t \rangle, f\langle t \rangle) = \mathcal{U}(n,m)(f\langle t \rangle, (f\langle 0 \rangle)\langle t \rangle) \subset \mathcal{U}^*(n,m)(f\langle t \rangle, f).$$

These isomorphisms $f \cong f\langle t \rangle$ make the Grothendieck group $K_0(\mathcal{U}^*)$ into a $\mathbb{Z}$-module, rather than $\mathbb{Z}[q, q^{-1}]$-module since $[f]_{\cong} = [f\langle t \rangle]_{\cong}$ in $\mathcal{U}^*$.

The horizontal composition in $\mathcal{U}$ induces horizontal composition in $\mathcal{U}^*$. It follows that the $\mathcal{U}^*(n,m)$, $n, m \in \mathbb{Z}$, form an additive 2-category.

The Karoubi envelope $\mathrm{Kar}(\mathcal{U}^*)$ will be denoted by $\dot{\mathcal{U}}^*$, which is equivalent as an additive 2-category to the additive 2-category obtained from $\dot{\mathcal{U}}$ by defining

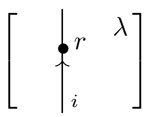
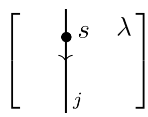
$$\dot{\mathcal{U}}^*(n,m)(f,g) = \bigoplus_{t\in\mathbb{Z}} \dot{\mathcal{U}}(n,m)(f, g\langle t\rangle).$$

5. A homomorphism from the current algebra

In type A a homomorphism from the current algebra to the trace of the 2-category $\mathcal{U}^*$ was constructed in [4] and [3]. For $\mathfrak{sl}_2$ is it proven to be an isomorphism for the integral version of $\mathcal{U}^*$ in [4].

The image $\mathsf{E}_{i,r}^{(a)}$ of the divided power of the current algebra generator is given by $[y_1^r \cdots y_a^r e_a]$, which is the rth power of a dot on each of a consecutive strands, multiplied by a primitive idempotent in the nilHecke algebra (see Proposition 9.7 and Corollary 9.8 in [4]). Since any two primitive idempotents are equal in the trace, and the identity is the sum of $a!$ many such idempotents, this indeed satisfies $\mathsf{E}_{i,r}^a = (a!)\mathsf{E}_{i,r}^{(a)}$.

Proposition 5.1. Let $\mathsf{E}_{i,r}1_\lambda, \mathsf{F}_{j,s}1_\lambda, \mathsf{H}_{i,r}1_\lambda$ denote the elements of $\mathrm{Tr}(\mathcal{U}_Q^*(\mathfrak{g}))$:

$\mathsf{E}_{i,r}1_\lambda :=$ , $\mathsf{F}_{j,s}1_\lambda :=$ , $\mathsf{H}_{i,r}1_\lambda := [p_{i,r}(\lambda)\,\mathrm{Id}_{\mathbf{1}_\lambda}]$,

where $p_{i,r}(\lambda)$ was defined in equation (4.26). For any choice of $(-)$-compatible choice of scalars Q arising from a coloring $c\colon \Gamma \to \mathbb{Z}_2$ of the graph Γ associated to the simply-laced Kac-Moody algebra $\mathfrak{g}$, there is a well defined homomorphism

$$\rho\colon \dot{\mathbf{U}}(\mathfrak{g}[t]) \longrightarrow \mathrm{Tr}(\mathcal{U}_Q^*(\mathfrak{g})), \tag{5.1}$$

given by

$$(x_{i,r}^+)^{(a)}1_\lambda \mapsto (-1)^{ac(i)\cdot r}\,\mathsf{E}_{i,r}^{(a)}1_\lambda, \qquad (x_{j,s}^-)^{(a)}1_\lambda \mapsto (-1)^{ac(j)\cdot s}\,\mathsf{F}_{j,s}^{(a)}1_\lambda, \qquad \xi_{i,r}1_\lambda \mapsto (-1)^{c(i)\cdot r}\,\mathsf{H}_{i,r}1_\lambda. \tag{5.2}$$

We will denote by $\rho^\pm$ and ρ^0 the restrictions of ρ to the subalgebras $\dot{\mathbf{U}}^\pm(\mathfrak{g}[t])$ and $\dot{\mathbf{U}}^0(\mathfrak{g}[t])$, respectively.

Proof. To prove this proposition we verify the current algebra relations using the relations in the 2-category $\mathcal{U}_Q^*(\mathfrak{g})$. We only need to consider the case $i \neq j$, since the relations in $\mathcal{U}_Q^*(\mathfrak{sl}_2)$ have been proven in [4]. **C1** is clear, since bubbles commute with each other. Axiom **C2** follows immediately from the definition of $p_{i,0}\mathbf{1}_\lambda = \langle i,\lambda\rangle \mathbf{1}_\lambda$.

Consider the equality **C3**. The case $a_{ij} = 0$ follows easily from the bubble slide relation. Suppose $a_{ij} = -1$. Using the power sum slide identity (4.27), we get

$$\mathsf{H}_{i,r}\mathsf{E}_{j,s}1_\lambda \;=\; \mathsf{E}_{j,s}\mathsf{H}_{i,r}1_\lambda - (-v_{ij})^r\,\mathsf{E}_{j,r+s}1_\lambda.$$

The relation

$$[\mathsf{H}_{i,r}, \mathsf{F}_{j,s}]1_\lambda = (-v_{ij})^r \mathsf{F}_{j,r+s}1_\lambda.$$

can be proven in a similar way. Since Q is a $(-)$-compatible choice of scalars arising from a coloring of the graph Γ, our color dependent rescaling in (5.2) gives axiom **C3**.

C4 follows from the relations (4.7) and (4.9), which imply

$$[\mathsf{E}_{i,r+1}, \mathsf{E}_{j,s}]1_\lambda = -v_{ij}[\mathsf{E}_{i,r}, \mathsf{E}_{j,s+1}]1_\lambda.$$

If we reverse the arrows in the preceding equation, they still hold:

$$[\mathsf{F}_{i,r+1}, \mathsf{F}_{j,s}]1_\lambda = -v_{ij}[\mathsf{F}_{i,r}, \mathsf{F}_{j,s+1}]1_\lambda.$$

The choice of signs in (5.2) corresponds to

$$[x^{\pm}_{i,r+1}, x^{\pm}_{j,s}] = [x^{\pm}_{i,r}, x^{\pm}_{j,s+1}].$$

The relation **C5** for the case $i \neq j$ follows from (4.12) which can be used to show

$$\mathsf{E}_{i,r}\mathsf{F}_{j,s}1_\lambda \;\;=\;\; \mathsf{F}_{j,s}\mathsf{E}_{i,r}1_\lambda.$$

To verify axiom **C6** in the case when $a_{ij} = -1$, the color dependent rescalings play no role. By a direct computation we get

$$\mathsf{E}_{i,r_1}\mathsf{E}_{j,s}\mathsf{E}_{i,r_2}1_\lambda + \mathsf{E}_{i,r_2}\mathsf{E}_{j,s}\mathsf{E}_{i,r_1}1_\lambda = \mathsf{E}_{i,r_1}\mathsf{E}_{i,r_2}\mathsf{E}_{j,s}1_\lambda + \mathsf{E}_{j,s}\mathsf{E}_{i,r_1}\mathsf{E}_{i,r_2}1_\lambda.$$

When $a_{ij} = 0$ axiom **C6** follows since

$$x^+_{i,r}x^+_{j,s} = (-1)^{c(i)r}(-1)^{c(j)s}\mathsf{E}_{i,r}\mathsf{E}_{j,s} = (-1)^{c(j)s}(-1)^{c(i)r}t_{ij}^{-1}t_{ji}\mathsf{E}_{j,s}\mathsf{E}_{i,r} = x^+_{i,r}x^+_{j,s},$$

since $t_{ij} = t_{ji}$ when $a_{ij} = 0$. The case $[x^-_{i,r}, x^-_{j,s}]$ is proven similarly. □

5.1. **Triangular decomposition.** Let $\mathcal{U}^+_{\mathrm{tr}} = \mathcal{U}^+_{\mathrm{tr},Q}$ denote the $\Bbbk$-linear subcategory of $\mathrm{Tr}(\mathcal{U}^*_Q(\mathfrak{g}))$ with objects indexed by the weight lattice $\mathrm{Ob}(\mathcal{U}^+_{\mathrm{tr}}) = X$, and with morphisms generated by composites of $\mathsf{E}_{i,a}$ for $i \in I$ and $a \geq 0$. Similarly, let $\mathcal{U}^-_{\mathrm{tr}} = \mathcal{U}^-_{\mathrm{tr,Q}}$ denote the $\Bbbk$-linear subcategory of $\mathrm{Tr}(\mathcal{U}^*_Q(\mathfrak{g}))$ with objects $\mathrm{Ob}(\mathcal{U}^-_{\mathrm{tr}}) = X$ and morphisms generated by $\mathsf{F}_{j,b}$ for $j \in I$ and $b \geq 0$. We define $\mathcal{U}^0_{\mathrm{tr}} = \mathcal{U}^0_{\mathrm{tr,Q}}$ as the $\Bbbk$-linear subcategory of $\mathrm{Tr}(\mathcal{U}^*_Q(\mathfrak{g}))$ with objects $\mathrm{Ob}(\mathcal{U}^0_{\mathrm{tr}}) = X$ and morphisms generated by bubbles. So we have $\mathcal{U}^0_{\mathrm{tr}} \cong \mathrm{Sym}^{|I|}$, where $|I|$ is the cardinality of I.

Proposition 5.2. $\mathcal{U}^0_{\mathrm{tr}}$ is isomorphic to $\dot{\mathbf{U}}^0(\mathfrak{g}[t])$.

Proof. For any $\Bbbk$ of characteristic 0, the map ρ restricted to $\dot{\mathbf{U}}^0(\mathfrak{g}[t])$ is an isomorphism, since power sums form a $\mathbb{Q}$-basis of symmetric functions. To show the result for a finite field $\Bbbk$, we first construct an isomorphism over $\mathbb{Z}$ and then tensor it with $\Bbbk$. For that we use Garland's integral basis of the current algebra defined in [19]. Since $\dot{\mathbf{U}}^0(\mathfrak{g}[t])$ is isomorphic to the tensor product of $|I|$ copies of $\dot{\mathbf{U}}^0(\mathfrak{sl}_2[t])$, we apply Lemma 8.2 in [4] to get an isomorphisms between Sym and Garland's $\mathbb{Z}$-basis of the Cartan part for each copy of $\mathfrak{sl}_2$. □

Proposition 5.3. Let $[f]$ be the class of a 2-endomorphism in $\mathrm{Tr}(\mathcal{U}^*(\mathfrak{g}))$. Then $[f]$ can be expressed as a sum

$$[f] = \sum [f^0][f^+][f^-]$$

where $f^{\pm}$, and f^0 are 2-endomorphisms in $\mathcal{U}^{\pm}_{\mathrm{tr}}$, $\mathcal{U}^0_{\mathrm{tr}}$, respectively.

Proof. Assume f is a 2-endomorphism of a sequence $\boldsymbol{i} \in I^{|\boldsymbol{i}|}$ of generating 1-morphisms $\mathcal{E}_i\mathbf{1}_\lambda$ and $\mathcal{F}_i\mathbf{1}_\lambda$ in $\mathcal{U}^*$ of length $|i|$. To make 1-morphisms to be determined by such sequences, we will use negative labels for $\mathcal{F}$'s, i.e. $\mathcal{E}_{-i}\mathbf{1}_\lambda := \mathcal{F}_i\mathbf{1}_\lambda$. We need to show that the trace class $[f]$ decomposes as a $\Bbbk$-linear combination of elements of the form $[f^0][f^+][f^-]$. Note that using bubble slide relations we can move all bubbles at any place of the diagram, so let us assume that they are at the far left in what follows.

We first claim that $[f]$ belongs to the span of 2-endomorphisms of $\mathcal{E}_{\boldsymbol{i}^+}\mathcal{E}_{\boldsymbol{i}^-}$, where $\boldsymbol{i}^+$ are all positive and $\boldsymbol{i}^-$ are all negative so that all $\mathcal{E}$'s are to the left of $\mathcal{F}$'s. We prove this by induction on the length $|\boldsymbol{i}|$ of $\boldsymbol{i}$ and the number of inversions needed to bring $\boldsymbol{i}$ into the form $\boldsymbol{i}^+\boldsymbol{i}^-$.

To illustrate the argument assume $|\boldsymbol{i}| = 2$ and the number of inversions is 1. If f contains cap and cup, then its trace class is in the span of bubbles. Let f be the identity morphism of $\mathcal{F}_i\mathcal{E}_j\mathbf{1}_\lambda$. Using (4.12) if $i \neq j$, and (4.19) if $i = j$, together with the trace relation we can decompose $[f]$ into an endomorphism of $\mathcal{E}_j\mathcal{F}_i$ plus terms with $|\boldsymbol{i}| = 0$. The same holds if f contains crossings. More generally, if f is any 2-endomorphism of $\mathcal{E}_{\boldsymbol{i}}\mathbf{1}_\lambda$, then if some $\mathcal{F}_j$ appears left of an $\mathcal{E}_i$ observe that using (4.12) or

(4.19) and trace relations as before, we can write $[f]$ as a sum of $[g]$ which has one inversion less than f and terms of length less that $|\boldsymbol{i}|$.

Hence, it remains to show that any endomorphism f of $\mathcal{E}_{\boldsymbol{i}}\mathbf{1}_\lambda = \mathcal{E}_{\boldsymbol{i}^+}\mathcal{E}_{\boldsymbol{i}^-}\mathbf{1}_\lambda$ is of the desired form. It is not hard to show that any such f can be written as a sum of diagrams containing no caps and cups using the trace condition and the relations in $\mathcal{U}^*$, so the result follows. □

6. Surjectivity results

6.1. Surjectivity of ρ.

Theorem 6.1. The natural map $\rho^-\colon \dot{\mathbf{U}}^-(\mathfrak{g}[t]) \to \mathcal{U}^-_{\mathrm{tr}}$ is a surjection.

Choose any infinite sequence $\mathbf{i} = (i_1, i_2, \dots) \in I^{\mathbb{Z}_{>0}}$ such that every element of I appears an infinite number of times. For any infinite word in the integers, $\mathbf{a} = (a_1, a_2, \dots)$ with almost all $a_i = 0$, we let $\mathbf{i_a}$ be the concatenated word $i_1^{a_1} i_2^{a_2} \cdots$, and let $\mathcal{F}_{\mathbf{a}}$ be the associated 1-morphism in $\mathcal{U}^-$.

Use induction over the lexicographic order on sequences to show that $\mathcal{U}^-_{\mathrm{tr}}$ is spanned by dots.

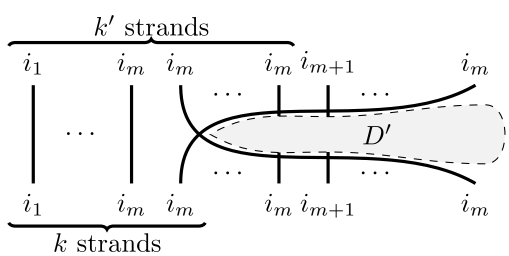
Proof. We'll give an inductive proof of the following statement:

($*_{\mathbf{a}}$) The image of $\mathrm{End}(\mathcal{F}_{\mathbf{a}})$ in $\mathcal{U}^-_{\mathrm{tr}}$ lies in the sum of images of the polynomial endomorphisms of $\mathrm{End}(\mathcal{F}_{\mathbf{b}})$ for $\mathbf{b} \geq \mathbf{a}$.

Since every 2-morphism in $\mathcal{U}^-$ factors through a finite sum of $\mathcal{F}_{\mathbf{a}}$'s, establishing this for every $\mathbf{a}$ will complete the proof.

First, assume that only one entry of $\mathbf{a}$ is non-zero. In this case, $\mathrm{End}(\mathcal{F}_{\mathbf{a}})$ is a nil-Hecke algebra, and thus has trace generated by its polynomial subalgebra, as proven in [4, 9.8].

Now assume that $\mathbf{a}$ is arbitrary. Every endomorphism of $\mathcal{F}_{\mathbf{a}}$ can be written as a sum of diagrams, so we may as well consider the case of a single diagram D. If the diagram has no crossings, it is polynomial, and we are done. Now having fixed $\mathbf{a}$, we induct on the number of crossings. Modulo elements with a lower number of crossings than D, we can isotope the strands of D though crossings. In particular, for some k, we can assume that the leftmost $k-1$ strands have no crossings, and that the strands which kth from the left at the top and bottom cross. Let us call these strands U and V. We can further assume that all crossings occur to the right of U and of V (or on the strands themselves), in the region marked D' in the diagram below. We'll now also induct (upward) on k.



Let m be the smallest integer such that $k' = a_1 + \cdots + a_m \geq k$. Note that by definition, the strands between the kth and k'th from the left are the same color by definition. Thus, if the strands U and V have both ends left of the k'th strand, we can get rid of their crossing, and increase k. Thus, we can assume that U and V do have one end at the kth terminal from left and their other at a terminal further right than the k'th.

We wish to show that D factors through $\mathcal{F}_{\mathbf{b}}$, where $\mathbf{b} = (a_1, a_2, \dots, a_m+1, \dots)$, which is thus higher in lexicographic order. Consider that after crossing V, the strand U crosses the $k+1$st strand from the left, the $k+2$nd, etc. until it reaches the k'th. Below U, these other strands don't cross; thus, they all have the same label at U, that is i_m. Thus, at the y-value just below the crossing of U and k'th strand, we see a_1 strands with label i_1, etc. up until a_m strands with label i_m, followed by U which also has this label. Thus, indeed, this slice is associated to $\mathbf{b} = (a_1, a_2, \dots, a_m + 1, \dots)$, which is, of course, greater in lexicographic order than $\mathbf{a}$.

Following through the induction on k, this shows $(*_\mathbf{a})$ and thus the desired statement. □

Theorem 6.2. The homomorphism $\rho\colon \dot{\mathbf{U}}(\mathfrak{g}[t]) \longrightarrow \mathrm{Tr}(\mathcal{U}_Q^*(\mathfrak{g}))$, is surjective for all $\mathfrak{g}$.

Proof. By the triangular decomposition of Proposition 5.3, it suffices to show that the image contains any class $[f^\pm]$ or $[f^0]$ where $f^\pm$, and f^0 are in $\mathcal{U}_{\mathrm{tr}}^\pm, \mathcal{U}_{\mathrm{tr}}^0$, respectively. For $[f^\pm]$, this follows immediately from Theorem 6.1. For f^0, this is clear by the isomorphism between $1_\lambda \dot{\mathbf{U}}^0(\mathfrak{g}[t]) 1_\lambda \cong \mathrm{End}(\mathbf{1}_\lambda) = \mathrm{Sym}^{|I|}$ for any weight λ. □

Note that even if Q is not a $(-)$-compatible choice of scalars, we can use the argument of Theorem 6.2 to show the weaker statement:

Corollary 6.3. For any symmetrizable Kac-Moody algebra $\mathfrak{g}$ and choice of scalars Q, the elements $\mathsf{E}_{i,r}^{(n)} 1_\lambda, \mathsf{F}_{i,r}^{(n)} 1_\lambda$ for $i \in I$, $r \geq 0$, $n \geq 1$ and $\lambda \in X$. generate $\mathrm{Tr}(\mathcal{U}_Q^*(\mathfrak{g}))$ as an algebra.

It would be quite interesting to obtain a uniform description of these algebras in terms of a Drinfeld-type presentation.

7. Injectivity results

7.1. **Cyclotomic quotients.** Fix a highest weight λ. The 2-category $\dot{\mathcal{U}}_Q^*$ has a principal representation $\dot{\mathcal{U}}_Q^*(\lambda, *)$ which sends the weight μ to the graded category of 1-morphisms $\lambda \to \mu$ as defined in [47].

We wish to consider two natural quotients of $\dot{\mathcal{U}}_Q^*(\lambda, *)$:

- $\check{\mathcal{U}}^\lambda$ is the quotient of $\dot{\mathcal{U}}_Q^*(\lambda, *)$ by the subrepresentation generated by $\mathbf{1}_{\lambda+\alpha_\mathbf{i}}$ for all $i \in I$. That is, we set to 0 any 2-morphism factoring through a 1-morphism of the form $A\mathcal{E}_i \,\mathrm{id}_\lambda$ for A arbitrary.
- $\mathcal{U}^\lambda$ is the quotient of $\check{\mathcal{U}}^\lambda$ by all positive degree endomorphisms of $\mathbf{1}_\lambda$.

These categories also have explicit realizations in terms of finite dimensional algebras.

Recall that the KLR algebra $R = R_Q$ from Definition 4.2 has an algebraic realization where $e(\boldsymbol{i})$ are idempotents corresponding to sequences $\boldsymbol{i} = (i_1, \dots, i_m)$, and $y_r e(\boldsymbol{i})$ denotes a dot on the rth strand. Let λ be a dominant weight, and recall the cyclotomic quotient R^λ from [24] defined as the quotient of R by the two sided ideal generated by the relations

$$\left\{ y_1^{\langle i,\lambda\rangle} e(\boldsymbol{i}) = 0 \mid \text{for all sequences } \boldsymbol{i}. \right\} \tag{7.1}$$

Likewise, we will also consider the the deformed cyclotomic quotient $\check{R}^\lambda$ defined in [47, 3.24]; this is a quotient of the usual KLR algebra by the relation

$$\left(y_1^{\langle i,\lambda\rangle} + q_1^{(i)} y^{\langle i,\lambda\rangle - 1} + \cdots + q_{\langle i,\lambda\rangle}^{(i)} \right) e(\mathbf{i}) = 0 \tag{7.2}$$

where each $q_k^{(i)}$ is a free deformation parameter of degree $2k$. We let R_μ^λ (respectively $\check{R}_\mu^\lambda$) denote the summand of this algebra categorifying the μ-weight space, that is, that where the labels on strands add up to $\lambda - \mu$. The categories of modules over both R^λ and $\check{R}^\lambda$ each have a categorical action of $\mathfrak{g}$, where each $\mathcal{F}_i$ is an induction functor and $\mathcal{E}_i$ a restriction functor [22, 47].

Theorem 7.1 ([47, 3.20, 3.25])**.** The category $\mathcal{U}^\lambda(\mu)$ (resp. $\check{\mathcal{U}}^\lambda(\mu)$) is equivalent to the projective modules over the ring $\check{R}_\mu^\lambda$ (resp. R_μ^λ). The Grothendieck groups of $\bigoplus_{\mu\leq\lambda} \mathcal{U}^\lambda(\mu)$ and $\bigoplus_{\mu\leq\lambda} \check{\mathcal{U}}^\lambda(\mu)$ are canonically isomorphic, and both isomorphic to the simple integrable representation $V(\lambda)$.

Note that this implies that these categorical modules are integrable in the sense of Chuang-Rouquier: any object M is killed by E_i^m or F_i^m for $m \gg 0$, since the resulting object lies in a trivial weight space.

Then we have a composite of surjections

$$\mathbf{U}(\mathfrak{g}[t]) \longrightarrow \mathrm{Tr}(\mathcal{U}(\mathfrak{g})) \longrightarrow \mathrm{Tr}(\check{\mathcal{U}}^\lambda(\mathfrak{g})) \longrightarrow \mathrm{Tr}(\mathcal{U}^\lambda(\mathfrak{g})).$$

Proposition 7.2. There are surjective homogeneous maps of $\mathbf{U}(\mathfrak{g}[t])$-modules

$$W(\lambda) \longrightarrow \mathrm{Tr}(\mathcal{U}^\lambda(\mathfrak{g})) \qquad \mathbb{W}(\lambda) \longrightarrow \mathrm{Tr}(\check{\mathcal{U}}^\lambda(\mathfrak{g}))$$

for all simply laced Kac-Moody algebras $\mathfrak{g}$ containing no odd cycles. (More generally, the result holds for any choice of scalars as in Proposition 5.1.)

Proof. The trace of $\mathcal{U}^\lambda(\mathfrak{g})$ or $\check{\mathcal{U}}^\lambda(\mathfrak{g})$ is an integrable representation of the current algebra $\mathbf{U}(\mathfrak{g}[t])$. This representation is generated as a $\mathfrak{g}[t]$-module by the trace of the empty diagram in weight λ, which is homogeneous of degree 0. Furthermore, the cyclotomic relation (7.1) or (7.2) implies that $\mathfrak{n}^+$ acts trivially on this vector. By the presentation [14, (3.5)], any integrable $\mathfrak{g}[t]$-module V generated by an element $v \in V$ satisfying the relations

$$\mathfrak{n}^+[t]v = 0, \qquad hv = \lambda(h)v \tag{7.3}$$

is a quotient of the global Weyl module $\mathbb{W}(\lambda)$. This shows the result for $\mathrm{Tr}(\check{\mathcal{U}}^\lambda(\mathfrak{g}))$. In $\mathcal{U}^\lambda(\mathfrak{g})$, by definition, all higher degree bubbles act trivially on v. Thus, the surjection of the global Weyl module factors through the quotient by the relation $\mathfrak{h}t[t]v = 0$. That is, V is a quotient of the local Weyl module $W(\lambda)$. □

7.2. Injectivity of ρ in types ADE.

Theorem 7.3. For $\mathfrak{g}$ of type ADE, the surjective map $W(\lambda) \longrightarrow \mathrm{Tr}(\mathcal{U}^\lambda(\mathfrak{g}))$ is an isomorphism.

Proof. Let $\lambda_{\min}$ be the unique minimal dominant weight amongst those $\leq \lambda$. If λ lies in the root lattice, then $\lambda_{\min} = 0$; if λ does not lie in the root lattice, then $\lambda_{\min}$ will be the unique highest weight of a minuscule representation in that coset of the root lattice. If $\mathfrak{g} = \mathfrak{sl}_n$, then $V(\lambda_{\min}) = \bigwedge^k \mathbb{C}^n$ where $0 \leq k < n$ is chosen so that the scalar matrix $e^{2\pi i/n} I \in \mathrm{SL}(n)$ acts by $e^{2\pi i k/n}$ on $V(\lambda)$. For $\mathfrak{g} = \mathfrak{so}_{2n}$, we have that

- $V(\lambda_{\min}) = \mathbb{C}$ is the trivial representation if $V(\lambda)$ is a representation of $\mathrm{SO}(2n)$ on which $-I \in \mathrm{SO}(2n)$ acts trivially,
- $V(\lambda_{\min}) = \mathbb{C}^{2n}$ is the vector representation if $V(\lambda)$ is a representation of $\mathrm{SO}(2n)$ on which $-I \in \mathrm{SO}(2n)$ acts by -1,
- $V(\lambda_{\min}) = S^\pm$ is one of the two half-spinor representations if $V(\lambda)$ is a representation of $\mathrm{Spin}(2n)$ not factoring through $\mathrm{SO}(2n)$ (determined by having the same action of the center of $\mathrm{Spin}(2n)$ as $V(\lambda)$).

By [26, 3.9], the socle filtration of the local Weyl module of type λ coincides with the degree filtration. In particular, by [26, 3.14], the socle itself is given by the homogeneous elements of degree $\langle \lambda, \lambda \rangle - \langle \lambda_m, \lambda_m \rangle$ and this is a single copy of the simple module $V(\lambda_m)$. By its universal property, any graded module M over the current algebra which is generated by a single highest weight element of weight λ and degree 0 receives a surjective map from the local Weyl module. Thus, if M also contains a non-zero element of degree $\langle \lambda, \lambda \rangle - \langle \lambda_{\min}, \lambda_{\min} \rangle$, this map is not zero on the socle of the Weyl module. Since the Weyl module has finite length, every submodule contains a simple submodule, which lies in the socle by definition. By [26, 3.8], the socle of $W(\lambda)$ is simple, so any non-zero submodule of $W(\lambda)$ contains $\mathrm{soc}(W(\lambda))$. However, the kernel of the map to M does not contain this submodule, and thus is 0.

By Proposition 7.2, $\mathrm{Tr}(\mathcal{U}^\lambda(\mathfrak{g}))$ is generated by such an element. Furthermore, using the isomorphism of Theorem 7.1, the symmetric Frobenius trace (described in [47, Rk. 3.19]) on the algebra $R^\lambda_{\lambda_{\min}}$

induces a non-zero functional on $\mathrm{Tr}(\mathcal{U}^\lambda(\mathfrak{g}))$ of degree $-\langle\lambda,\lambda\rangle+\langle\lambda_{\min},\lambda_{\min}\rangle$, which shows that this space has non-zero elements of degree $\langle\lambda,\lambda\rangle-\langle\lambda_{\min},\lambda_{\min}\rangle$. Thus, we must have the desired isomorphism. □

The map $\mathbb{W}(\lambda) \longrightarrow \mathrm{Tr}(\check{\mathcal{U}}^\lambda(\mathfrak{g}))$ induces a surjective homogeneous ring homomorphism $\mathbb{A}_\lambda \to \check{R}^\lambda_\lambda$. By the definition [47, 3.24], $\check{R}^\lambda_\lambda$ is a polynomial ring over the fake bubbles. Thus, these rings have the same Hilbert series, and this map must be an isomorphism.

Theorem 7.4. For $\mathfrak{g}$ of type ADE, the surjective map $\mathbb{W}(\lambda) \longrightarrow \mathrm{Tr}(\check{\mathcal{U}}^\lambda(\mathfrak{g}))$ is an isomorphism.

Proof. The induced map is an isomorphism modulo the unique maximal homogeneous ideal by Theorem 7.3. At a generic maximal ideal, the specialization of $\mathbb{W}(\lambda)$ is isomorphic to a tensor product of shifted local Weyl modules for fundamental weights, with ω_i appearing with multiplicity $m_i = \langle i,\lambda\rangle$ by [14, 5.8]. Similarly, the specialization of $\check{R}^\lambda$ is a tensor product of shifted cyclotomic quotients for fundamental weights with R^{ω_i} appearing with multiplicity m_i by [44, 2.23]. Thus, the same is true on the level of traces. Theorem 7.3 applied to the fundamental weights shows that these modules over $\mathbb{A}_\lambda \cong \check{R}^\lambda_\lambda$ have the same generic rank. Thus, any surjective map from one to the other is necessarily an isomorphism. □

8. Trace categorification results

8.1. A trace decategorification of $\mathcal{U}_Q(\mathfrak{g})$.

Theorem 8.1. The Chern character map $h_{\mathcal{U}_Q(\mathfrak{g})}\colon K_0(\dot{\mathcal{U}}_Q(\mathfrak{g}))\otimes_{\mathbb{Z}} \Bbbk \cong \dot{\mathbf{U}}_q(\mathfrak{g}) \to \mathrm{Tr}(\mathcal{U}_Q(\mathfrak{g}))$ is an isomorphism.

Proof. By Lemma 2.4, this map is injective. On the other hand, the elements $\mathsf{E}^{(n)}_{i,0}1_\lambda, \mathsf{F}^{(n)}_{j,0}1_\lambda$ are the only degree 0 elements of the generating set of Corollary 6.3, with all others of positive degree. Thus, they must generate $\mathrm{Tr}(\mathcal{U}_Q(\mathfrak{g}))$. The elements $\mathsf{E}^{(n)}_{i,0}1_\lambda, \mathsf{F}^{(n)}_{j,0}1_\lambda$ are given by idempotents, and thus obviously in the image of $h_{\mathcal{U}_Q(\mathfrak{g})}$. This shows the map is surjective as well. □

The special cases of Theorem 8.1 for $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3$ are proved in [4, 48], where the Chern character maps are defined over $\mathbb{Z}$.

Remark 8.2. The notion of a strongly upper-triangular category was defined in [4, Section 4.1]. Such categories possess a distinguished basis of objects B and the results of [4, Proposition 4.6] imply that $\mathrm{Tr}(\mathcal{C}) = \mathrm{HH}_0(\mathcal{C}) \cong \Bbbk B$ and that all higher Hochschild homology vanishes, $\mathrm{HH}_i(\mathcal{C}) = 0$, for $i > 0$. It follows from results in [46] that the basis of indecomposables in $\mathcal{U}_Q(\mathfrak{g})$ is strongly upper-triangular, if $\Bbbk$ has characteristic 0 and $t_{ij}t_{ji} = -1$ whenever $\langle\alpha_i,\alpha_j\rangle = -1$. Hence, Theorem 8.1 can be extended to include the fact that $\mathrm{HH}_i(\mathcal{U}_Q(\mathfrak{g})) = 0$ for $i > 0$ under the same hypotheses.

8.2. A trace decategorification of $\mathcal{U}^*_Q(\mathfrak{g})$.

Theorem 8.3. For any choice of $(-)$-compatible choice of scalars Q arising from a coloring $c\colon \Gamma \to \mathbb{Z}_2$ of the graph Γ associated to the simply-laced Kac-Moody algebra $\mathfrak{g}$ of type ADE, the homomorphism

$$\rho\colon \dot{\mathbf{U}}(\mathfrak{g}[t]) \longrightarrow \mathrm{Tr}(\mathcal{U}^*_Q(\mathfrak{g})), \tag{8.1}$$

is an isomorphism.

Proof. Lemma 3.5 implies that the map ρ must be injective, since any element of its kernel would kill all global Weyl modules. Combining with Theorem 6.2 completes the proof. □

9. An action on centers of 2-representations

9.1. **Cyclic 2-categories and the center.** Given a linear 2-category $\mathbf{C}$ we define the center $Z(\lambda)$ of an object $\lambda \in \mathrm{Ob}(\mathbf{C})$ as the commutative ring of endomorphisms $\mathbf{C}(1_\lambda, 1_\lambda)$, see [18]. Note that in the linear 2-category **AdCat** of additive categories, additive functors, and natural transformations, the center $Z(\mathcal{C})$ of an object $\mathcal{C}$ is the endomorphism ring of the identity functor $\mathrm{Id}_{\mathcal{C}}$ on $\mathcal{C}$. Define the **center of objects** of the 2-category $\mathbf{C}$ as the $Z(\mathbf{C}) = \bigoplus_{\lambda \in \mathrm{Ob}(\mathbf{C})} Z(\lambda)$.

There is a fairly general framework under which the trace of a linear 2-category $\mathbf{C}$ acts on the center of objects $Z(\mathcal{K})$ of any 2-representation $F\colon \mathbf{C} \to \mathcal{K}$. This happens whenever the 2-category $\mathbf{C}$ has enough "coherent" duality. This idea is captured by the notion of a *cyclic 2-category* [17, 27]. In a cyclic 2-category $\mathbf{C}$ every one morphism $x\colon \lambda \to \lambda'$ is equipped with a specified biadjoint morphism $x^*\colon \lambda' \to \lambda$ and 2-morphisms

$$\mathrm{ev}_x\colon x^*x \to 1_\lambda \qquad \mathrm{coev}_x\colon 1_{\lambda'} \to xx^*$$
$$\widetilde{\mathrm{ev}}_x\colon xx^* \to 1_{\lambda'} \qquad \widetilde{\mathrm{coev}}_x\colon 1_\lambda \to x^*x$$

satisfying the adjunction axioms. Then given a 2-morphism $f\colon x \to y$ in $\mathbf{C}$ we can define the left and right dual of f:

$$f^* := (\mathrm{ev}_y \otimes \mathrm{Id}_{x^*})(\mathrm{Id}_y \otimes f \otimes \mathrm{Id}_{x^*})(\mathrm{Id}_{y^*} \otimes \mathrm{coev}_x)\colon y^* \to x^*$$
$${}^*f := (\mathrm{Id}_{x^*} \otimes \widetilde{\mathrm{ev}}_y)(\mathrm{Id}_{x^*} \otimes f \otimes \mathrm{Id}_{y^*})(\widetilde{\mathrm{coev}}_x \otimes \mathrm{Id}_{y^*})\colon y^* \to x^*.$$

(Here we are using $\otimes$ to denote the horizontal composition of 2-morphisms.) A 2-category $\mathbf{C}$ where all 1-morphisms have specified biadjoints is said to be *cyclic with respect to the biadjoint structure*, or simply a cyclic 2-category [3], when the left and right dual agree $f^* = {}^* f$, or equivalently $f^{**} = f$.

For $x\colon \lambda \to \lambda'$ a 1-morphism in $\mathbf{C}$ and $f\colon x \to x$ a 2-endomorphism in $\mathbf{C}$ representing a class $[f]$ in $\mathrm{Tr}(\mathbf{C})$, then $[f]$ defines an operator $Z(F(\lambda)) \to Z(F(\lambda'))$ sending the element $c\colon 1_{F(\lambda)} \to 1_{F(\lambda)}$ to the element given by the composite

$$\widetilde{\mathrm{ev}}_{F(x)} \circ (F(f) \otimes c \otimes \mathrm{Id}_{F(x^*)}) \circ \mathrm{coev}_{F(x)}\colon 1_{F(\lambda')} \to 1_{F(\lambda')} \in Z(\lambda'). \tag{9.1}$$

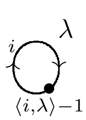
The following proposition is immediate.

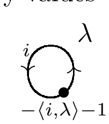
Proposition 9.1. A 2-representation $F\colon \mathbf{C} \to \mathcal{K}$ from a cyclic 2-category $\mathbf{C}$ into an linear 2-category $\mathcal{K}$ induces an action of $\mathrm{Tr}(\mathbf{C})$ on the center of objects $Z(\mathcal{K})$ given by (9.1).

In terms of graphical calculus, an element $c\colon 1_\lambda \to 1_\lambda$ of $Z(\lambda)$ can be represented by a closed diagram in weight λ. A class $[f]$ is represented by a diagram on an annulus with interior region labelled λ and exterior region labelled λ'. The action of $[f]$ on c is given by forgetting the annulus and placing the diagram for c into the interior region.

9.2. **Cyclicity for the 2-category $\mathcal{U}_Q(\mathfrak{g})$.** The Q-cyclic relation 4.4 implies that whenever $t_{ij} \neq 1$ the 2-category $\mathcal{U}_Q(\mathfrak{g})$ is not cyclic. However, in this section we will show that it is often possible to rescale this 2-category so that it is cyclic and Proposition 9.1 applies.

Using rescaling 2-functors defined in [28] it is possible to rescale the 2-category $\mathcal{U}_Q(\mathfrak{g})$ so that the values of i-colored degree zero bubbles take arbitrary values

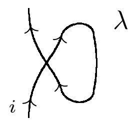
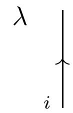
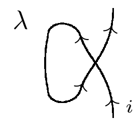
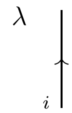


$= c^+_{i,\lambda}\mathrm{Id}_{\mathbf{1}_\lambda}$ for $\langle i, \lambda\rangle \geq 1$,


$= c^-_{i,\lambda}\mathrm{Id}_{\mathbf{1}_\lambda}$ for $\langle i, \lambda\rangle \leq -1$,

[3] A cyclic 2-category is the same thing as a pivotal 2-category [35]. This can be seen as a many object version of a pivotal monoidal category, see [20] where traces in this context are studied. Müger points out in [35, page 11] a strict pivotal 2-category can be defined from Mackaay's work [31] on spherical 2-categories by ignoring the monoidal structure.

and i-labelled curls in weights λ with $\langle i,\lambda\rangle = 0$ satisfy

(9.2)  $= \ -c^{+}_{i,\lambda}$   $= \ c^{-}_{i,\lambda}$  .

Consistency of the graphical calculus requires that these coefficients satisfy

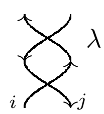
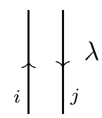
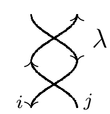
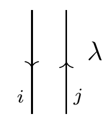
$$c^{+}_{i,\lambda+\alpha_j} = c^{+}_{i,\lambda}, \qquad c^{-}_{i,\lambda-\alpha_j} = c_{i,\lambda}, \qquad \text{for all } i,j \in I,$$

and

$$c^{+}_{i,\lambda}c^{-}_{i,\lambda} = 1 \quad \text{when } \langle i,\lambda\rangle = 0.$$

In particular, the coefficients $c^{\pm}_{i,\lambda}$ only depend on $\langle i,\lambda\rangle$ and are completely determined by the coefficients with $\langle i,\lambda\rangle = 0$ and $\langle i,\lambda\rangle = 1$, which we denote by $c^{+}_{i,0}$ and $c^{+}_{i,1}$, respectively.

The rescaling 2-functors are given by a weight dependent rescaling of caps and cups. They fix all sl_2 and KLR-relations. The mixed relations (4.12) change in the following way,

(9.3)  $= \ \dfrac{c_{j,\lambda}}{c_{j,\lambda+\alpha_i}} t_{ji}$   $= \ \dfrac{c_{i,\lambda}}{c_{i,\lambda+\alpha_j}} t_{ij}$ 

and the Q-cyclic relation 4.4 for crossings also depend on the parameters $c^{+}_{i,0}$ and $c^{+}_{i,1}$. Values of these parameters can be chosen so that the Q-cyclic relation becomes the usual cyclicity relation if and only if the equation

$$v_{ij} = c^{+}_{i,0}c^{+}_{i,1}c^{+}_{j,0}c^{+}_{j,1} \tag{9.4}$$

holds for all pairs $i,j \in I$ with $(\alpha_i,\alpha_j) = -1$. Note that when $(\alpha_i,\alpha_j) \neq 1$ cyclicity already holds.

Proposition 9.2. Let Γ be the graph associated to a simply laced Kac-Moody algebra. Suppose that Q is an integral choice of scalars arising from the $(+)$-compatible choice of scalars associated to a coloring of the graph Γ. Then the rescaling

$$c^{+}_{i,0} = 1, \qquad c^{+}_{i,1} = (-1)^{c(i)} \qquad \text{for all } i \in I,$$

makes the 2-category $\mathcal{U}_Q(\mathfrak{g})$ a cyclic 2-category.

Remark 9.3. Only the case of Γ with odd cycles all chosen with scalars $v_{ij} = -1$ is excluded by the $(+)$-compatibility requirement.

Proof. It is evident that the rescaling satisfies equation (9.4). □

Corollary 9.4. Let Γ be the graph associated to a simply laced Kac-Moody algebra. Suppose that Q is an integral choice of scalars arising from the $(+)$-compatible choice of scalars associated to a coloring of the graph Γ. Then $\mathrm{Tr}(\mathcal{U}_Q(\mathfrak{g}))$ acts on the center of objects $Z(\mathcal{K})$ in any 2-representation $\mathcal{U}_Q(\mathfrak{g}) \to \mathcal{K}$. Under this action, $Z(\mathcal{U}^{\lambda}(\mathfrak{g}))$ and $Z(\check{\mathcal{U}}^{\lambda}(\mathfrak{g}))$ can be identified with the dual local and global Weyl modules, respectively.

Proof. Only the last statement needs a proof. By Theorem 3.18 in [47], $\mathrm{Tr}(\mathcal{U}^{\lambda})$ is Frobenius, the non-degenerate pairing induces an isomorphism between 0th homology and 0th cohomology of the cyclotomic quotients. Moreover, the action of current algebra on the trace induces the adjoint action on the center, where the adjoint of $\mathsf{E}_i 1_{\lambda}$ is $1_{\lambda}\mathsf{F}_i$ up to degree shift, compare Definition 1.2 in [15]. In particular, the empty diagram in the highest weight cogenerates the center, in the sense that any submodule contains it. The result follows. □

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